A convenient approach to grading in homotopy theories

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Abstract

In this note we setup a convenient framework for considering graded homotopy groups in general homotopy theories. This is essentially expository, as most of this is implicit in the literature.

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1 Introduction

In an (in)famous rant, Adams highlights how grading in equivariant homotopy theory is a hairy business [Ada06, Section 6]. Folklore says that one should grade over the invertible *objects*. However, in [Dug14], Dugger gave a precise treatment of grading that shows this folklore is not so sound. In particular, Dugger shows that although one can choose a coherent collection of representations and isomorphisms that keep track of braidings, the resulting ring structure can depend on the choice of isomorphisms. The aim of this note is to show that if we instead grade over the *groupoid* of invertible objects, this business is not so hairy after all.

1.1 Motivating example

Let R_* be a $\mathbb{Z}$-graded ring. Experience tells us that the correct notion of *graded commutativity* is to introduce signs depending on the degree: R_* is graded commutative if for any homogeneous elements x and y we have

$$x \cdot y = (-1)^{\deg(x)\deg(y)} y \cdot x.$$

The introduction of this sign is typically justified by saying that $(-1)^{\deg(x)\deg(y)}$ is the degree of the relevant swap map. However, for at least two reasons, this is not an entirely satisfactory way to define a graded-commutative ring:

- This definition reads like a 'graded-ring object in abelian groups'. This makes it so that all the usual constructions we have for commutative algebra objects have to be altered. Instead, it would be preferable to have a definition that reads like a 'commutative algebra object in graded abelian groups'.
- It is not clear how to generalize this definition to other settings.[1]

Let's deal with the first objection. One would like to say that a $\mathbb{Z}$-graded-commutative ring is a commutative algebra object in $\mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$. Doing this requires endowing $\mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$ with a symmetric monoidal structure. Using Day convolution, it suffices to give symmetric monoidal structures on $\mathbb{Z}$ and Ab. For Ab we take the usual tensor product. However, there are many possible ways to make $\mathbb{Z}$ into a symmetric monoidal category. We consider $\mathbb{Z}$ as a category with objects integers, and the only non-empty hom sets are $\mathrm{Hom}(n,n)$ which are defined as $\mathrm{Aut}(\mathbb{Z})$ with composition given by the group structure on $\mathrm{Aut}(\mathbb{Z})$. To get the correct notion graded commutativity above we must introduce signs into the symmetric monoidal structure on $\mathbb{Z}$. The symmetric monoidal structure is given by $n \otimes_{\mathrm{Aut}(\mathbb{Z})} m = n + m$, and choosing the braiding $n \otimes_{\mathrm{Aut}(\mathbb{Z})} m \cong m \otimes_{\mathrm{Aut}(\mathbb{Z})} n$ to be $(-1)^{nm} \in \mathrm{Aut}(\mathbb{Z})$.

$$\begin{tikzcd} \cdots & \bullet \arrow[loop above, "\mathrm{Aut}(\mathbb{Z})"] & \bullet \arrow[loop above, "\mathrm{Aut}(\mathbb{Z})"] & \bullet \arrow[loop above, "\mathrm{Aut}(\mathbb{Z})"] & \cdots \end{tikzcd}$$

To avoid abuse of notation, we denote this symmetric monoidal 1-category by $(\mathbb{Z}, \otimes_{\mathrm{Aut}(\mathbb{Z})})$. Now we have a nice symmetric monoidal category, in which we could consider com-

[1]For example, if one tries to generalize this to an an equivariant setting by grading over $RO(G)$, and defining graded commutativity via degrees of swap maps $S^V \wedge S^W \to S^W \wedge S^V$, one immediately encounters issues from choosing isomorphism classes of V and W. Moreover, the degree of this swap map is an unit in the Burnside ring. Trying to define an $RO(G)$-graded abelian group as a sequence of abelian groups is too naive; the abelian groups must admit actions by $A(G)^\times$.

mutative algebra objects[2]. However, it is still not clear how to generalize this to other settings.

Question 1.1. *Can we rephrase the definition of a graded commutative ring so that our hand is forced, and we do not need to make a choice? In particular, does the category* $(\mathbb{Z}, \otimes_{\mathrm{Aut}(\mathbb{Z})})$ *appear in a more natural way?*

Answer 1.2. *Yes. The category* $(\mathbb{Z}, \otimes_{\mathrm{Aut}(\mathbb{Z})})$ *can be obtained as a* permutative skeleton *of the homotopy category of* $\mathcal{P}\mathrm{ic}(\mathrm{Sp})$*. In particular,* $(\mathbb{Z}, \otimes_{\mathrm{Aut}(\mathbb{Z})})$ *is equivalent as a symmetric monoidal category to the homotopy category of* $\mathcal{P}\mathrm{ic}(\mathrm{Sp})$*.*

Motivated by Equation (1.2), in this note we setup grading over a *groupoid* of invertible objects, and also how to reduce to a suitable skeleton when one wants to be explicit. Although, at first, grading over a groupoid and reducing to a skeleton seems obtuse, this is essentially implicit in the way one usually handles graded objects:

- As mentioned already, the Kozul sign rule is usually justified by saying $(-1)^{nm}$ it is the degree of the swap map $\mathbb{S}^n \otimes \mathbb{S}^m \to \mathbb{S}^m \otimes \mathbb{S}^n$. This is, of course, not precisely true. The degree is only defined for maps $\mathbb{S}^k \to \mathbb{S}^k$. Implicitly, when one says the 'degree of the swap map $\mathbb{S}^n \otimes \mathbb{S}^m \to \mathbb{S}^m \otimes \mathbb{S}^n$', one really means the degree of
$$\mathbb{S}^{n+m} \simeq \mathbb{S}^n \otimes \mathbb{S}^m \to \mathbb{S}^m \otimes \mathbb{S}^n \simeq \mathbb{S}^{n+m},$$
where one has fixed equivalences $\mathbb{S}^{n+m} \simeq \mathbb{S}^n \otimes \mathbb{S}^m$ and $\mathbb{S}^{n+m} \simeq \mathbb{S}^m \otimes \mathbb{S}^n$. Choosing equivalences in this manner amounts to taking a skeleton of the homotopy category of $\mathcal{P}\mathrm{ic}(\mathrm{Sp})$.

- Let X be a homotopy ring spectrum. Consider two maps $f : \mathbb{S}^n \to X$ and $g : \mathbb{S}^m \to X$. One defines the product of f and g as the composition
$$\mathbb{S}^n \otimes \mathbb{S}^m \xrightarrow{f \otimes g} X \otimes X \to X.$$
In this way, fg and gf live in different degrees. In order to compare fg and gf one fixes equivalence $\mathbb{S}^{n+m} \simeq \mathbb{S}^n \otimes \mathbb{S}^m$ and $\mathbb{S}^{n+m} \simeq \mathbb{S}^m \otimes \mathbb{S}^n$. Again, doing this precisely corresponds to taking a skeleton of the homotopy category of $\mathcal{P}\mathrm{ic}(\mathrm{Sp})$.

1.2 Reducing to a skeleton

By [JO12, Theorem B], it turns out that a particular nice skeleton can always be found. We review some definitions to be precise what we mean about a nice skeleton.

[2]We warn the reader that there are more commutative algebra objects in $\mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$ than there are things that would usually be considered graded-commutative groups. An object in $\mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$ has actions of $\mathrm{Aut}(\mathbb{Z})$. This subtlety is discussed later??

Definition 1.3. Let $\mathcal{G}$ be a symmetric monoidal 1-category. We say that $\mathcal{G}$ is a *Picard groupoid* is every object in $\mathcal{G}$ is invertible.

Example 1.4. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. Let $\mathcal{P}\mathrm{ic}(\mathcal{C})$ be the sub-∞-groupoid on the invertible objects in $\mathcal{C}$. Denote by $\mathrm{Pic}(\mathcal{C})$ the homotopy category of $\mathcal{P}\mathrm{ic}(\mathcal{C})$. Then $\mathrm{Pic}(\mathcal{C})$ is a Picard groupoid.

Definition 1.5. Let $\mathcal{G}$ and $\mathcal{I}$ be symmetric monoidal 1-categories. Let $\mathcal{I} \to \mathcal{G}$ be a functor. Suppose that $\mathcal{I}$ is skeletal. We say that $\mathcal{I}$ is a *permutative skeleton* of $\mathcal{G}$ if the symmetric monoidal structure of $\mathcal{I}$ is strictly associative, strictly unital, and is such that the functor $\mathcal{I} \to \mathcal{G}$ is symmetric monoidal.

Theorem 1.6 ([JO12, Theorem B])**.** *Any Picard groupoid admits a permutative skeleton.*

Example 1.7. The category $(\mathbb{Z}, \otimes_{\mathrm{Aut}(\mathbb{Z})})$ with $\mathbb{Z} \to \mathrm{Pic}(\mathrm{Sp})$ given by $n \mapsto \mathbb{S}^n$ is a permutative skeleton of $\mathrm{Pic}(\mathrm{Sp})$.

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2 General setup

Motivated by Equation (1.7), we consider $\mathrm{Pic}(\mathcal{C})$-gradings for suitable homotopy theories $\mathcal{C}$. It is useful to work slightly more generally and consider grading over $\mathcal{I}$, where $\mathcal{I}$ is a symmetric monoidal 1-category equipped with a suitable functor $\mathcal{I} \to \mathrm{Pic}(\mathcal{C})$. We work in the following general setup:

- $\mathcal{C}$ is a stable ∞-category with a closed symmetric monoidal structure. The symmetric monoidal unit in $\mathcal{C}$ is denoted by $\mathbf{1}$. The internal hom in $\mathcal{C}$ is denoted by $\mathrm{map}_{\mathcal{C}}(-,-)$. The internal hom in the homotopy category is denoted by $[-,-]$.
- $\mathcal{P}\mathrm{ic}(\mathcal{C})$ is the sub-∞-groupoid on the invertible objects in $\mathcal{C}$, and $\mathrm{Pic}(\mathcal{C})$ is the homotopy category of $\mathcal{P}\mathrm{ic}(\mathcal{C})$.
- $\mathcal{I}$ is a symmetric monoidal 1-category equipped with a symmetric monoidal functor $\mathcal{I} \to \mathrm{Pic}(\mathcal{C})$[3].

[3]The role of $\mathcal{I}$ is to enable us to grade over more manageable categories than $\mathcal{P}\mathrm{ic}(\mathcal{C})$.

- $\mathcal{A}$ is a symmetric monoidal 1-category such that the symmetric monoidal structure commutes with colimits in both variables separately. Typically $\mathcal{A} = \mathrm{Ab}$.
- $\pi{:}\mathcal{C} \to \mathcal{A}$ is a lax symmetric monoidal functor. Typically $\pi = [\mathbf{1}, -]$.
- All functor categories are given the Day convolution symmetric monoidal structure.

2.1 Graded homotopy groups

Our goal is to define a suitable notion of graded homotopy groups in $\mathcal{C}$. This should be a functor

$$\pi_\star{:}\mathcal{C} \to \mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathcal{A}).$$

Since $\mathcal{I}$ and $\mathcal{A}$ are symmetric monoidal, we may endow the functor category with the Day convolution symmetric monoidal structure. Hence we may ask for the graded homotopy groups functor to be lax symmetric monoidal.

Constructing a lax symmetric monoidal functor between ∞-categories by hand is hard work. Instead, we will define this as a composition of functors that are readily seen to be lax symmetric monoidal.

Lemma 2.1. *There is a lax symmetric monoidal functor*

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{P}\mathrm{ic}(\mathcal{C})^{\mathrm{op}}, \mathcal{C})$$

given on objects by $X \mapsto (L \mapsto \mathrm{map}_C(L, X))$ *where* $\mathrm{map}_{\mathcal{C}}(-,-)$ *is the internal hom in* $\mathcal{C}$*.*

Proof. By the universal property[4] of the Day convolution [Lur17, Construction 2.2.6.7], it suffices to give the data of a lax symmetric monoidal functor

$$\mathcal{C} \times \mathcal{P}\mathrm{ic}(\mathcal{C})^{\mathrm{op}} \to \mathcal{C}.$$

We will define such a functor as a composite functors that are known to be well defined functors between ∞-categories admitting lax symmetric monoidal structures.

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{P}\mathrm{ic}(\mathcal{C})^{\mathrm{op}} \arrow[r, dotted] \arrow[d] & \mathcal{C} \\
\mathcal{P}\mathrm{ic}(\mathcal{C})^{\mathrm{op}} \times \mathcal{C} \arrow[d, "\iota\times\mathrm{id}_{\mathcal{C}}"] & \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[uur, bend right, "\mathrm{map}_{\mathcal{C}}(-,-)"'] &
\end{tikzcd}$$

[4] For completeness, we spell out the universal property in Equation (A.2)

The first functor swaps the factors. In the second functor, $\iota:\mathcal{P}\mathrm{ic}(\mathcal{C})^{\mathrm{op}} \hookrightarrow \mathcal{C}^{\mathrm{op}}$ is given by the inclusion of $\mathcal{P}\mathrm{ic}(\mathcal{C})$ as a (not full) subcategory of $\mathcal{C}$. Since $\mathcal{P}\mathrm{ic}(\mathcal{C})$ is closed under the symmetric monoidal structure, this inclusion is a lax symmetric monoidal functor. The third functor is lax symmetric monoidal by [HHLN23, Cor 3.4.10]. Since the composition of lax symmetric monoidal functors is lax symmetric monoidal, we are done. □

Lemma 2.2. *Post-composition with $\pi:\mathcal{C} \to \mathcal{A}$ induces a lax symmetric monoidal functor*

$$\mathrm{Fun}(\mathcal{P}\mathrm{ic}(C)^{\mathrm{op}},\mathcal{C}) \to \mathrm{Fun}(\mathcal{P}\mathrm{ic}(C)^{\mathrm{op}},\mathcal{A}).$$

Proof. [Nik16, Cor. 3.7] □

Lemma 2.3. *Restriction along $\mathcal{I} \to \mathcal{P}\mathrm{ic}(\mathcal{C})$ induces a lax symmetric monoidal functor*

$$\mathrm{Fun}(\mathcal{P}\mathrm{ic}(C)^{\mathrm{op}},\mathcal{A}) \to \mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathcal{A}).$$

Proof. [HY19, Prop. A.4] □

Definition 2.4. We define the *graded homotopy groups* functor

$$\pi_\star:\mathcal{C} \to \mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathcal{A})$$

as the composition of the functors in Equations (2.1) to (2.3). For any $X \in \mathcal{C}$ and $l \in \mathcal{I}$, we denote the value of $\pi_\star(X)$ at l by $\pi_l(X)$.

Since $\pi_\star$ is as a composition of lax symmetric monoidal functors, the following is immediate.

Lemma 2.5. *The graded homotopy groups functor $\pi_\star$ is lax symmetric monoidal. Therefore, it sends $\mathcal{O}$-algebras in $\mathcal{C}$ to $\mathcal{O}$-algebras in $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathcal{A})$ for any operad $\mathcal{O}$. In particular, it sends commutative algebra objects to commutative algebra objects.*

Remark 2.6. Since $\mathcal{A}$ is (the nerve of) a 1-category, $\pi_\star$ sends $\mathbb{E}_2$-algebras in $\mathcal{C}$ to commutative algebra object in $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathcal{A})$. In fact, $\pi_\star$ sends commutative algebra objects in the homotopy category of $\mathcal{C}$ to commutative algebra objects in $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}},\mathcal{A})$.

Remark 2.7. By grading over a groupoid, we automatically keep track of the morphisms that often cause confusing sign issues. Essentially for free we get that graded homotopy groups are lax symmetric monoidal.

2.2 Examples

In this subsection, we give many examples of interest. First, we clarify the role of $\mathcal{I}$, and describe a general way to produce good choices for $\mathcal{I}$.

An obvious choice for $\mathcal{I}$ is to take $\mathcal{I} = \mathrm{Pic}(\mathcal{C})$, and $\mathcal{I} \to \mathrm{Pic}(\mathcal{C})$ to be the identity. This gives a notion of $\mathrm{Pic}(\mathcal{C})$-graded homotopy groups. These have good formal properties, and can be manipulated well. However, as illustrated by the following example, grading over the full groupoid can feel unfamiliar at first.

Example 2.8. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category with a closed symmetric monoidal structure. Take $\pi: \mathcal{C} \to \mathcal{A}$ to be $[\mathbf{1}, -]: \mathcal{C} \to \mathrm{Ab}$ where $\mathbf{1}$ is the symmetric monoidal unit, and $[-,-]$ is the internal hom in homotopy category of $\mathcal{C}$. Take $\mathcal{I} \to \mathrm{Pic}(\mathcal{C})$ to be the identity $\mathrm{Pic}(\mathcal{C}) \to \mathrm{Pic}(\mathcal{C})$. Now, for any $X \in \mathrm{CAlg}(\mathcal{C})$, it follows that $\pi_\star(X)$ is a commutative algebra object. Fix $l, l' \in \mathcal{P}\mathrm{ic}(\mathcal{C})$, and consider $x \in \pi_l(X)$ and $y \in \pi_{l'}(X)$. Typically, one expects that xy and yx live in the same group, and furthermore, one expects that xy and yx differ by a 'sign'. However, when grading over $\mathcal{P}\mathrm{ic}(\mathcal{C})$, the products xy and yx live in different degrees. They are however still related by a map. That is, if $\tau: l \otimes l' \to l' \otimes l$ is the braiding in $\mathcal{P}\mathrm{ic}(\mathcal{C})$ at l and l', then we have

$$xy = \tau^* yx$$

where τ^* is the image of τ under $\pi_\star$.

There is, however, a choice of $\mathcal{I}$ that feels more familiar. Note that the unfamiliar behavior in the above example comes from the fact that the objects $l \otimes l'$ and $l' \otimes l$ are not identical. Following Equation (1.7), we may alleviate this by taking $\mathcal{I}$ to be a permutative skeleton of $\mathrm{Pic}(\mathcal{C})$. The following general example can be further specialized to yield several familiar examples

Example 2.9. Let $\mathcal{C}$ be any stable ∞-category with a closed symmetric monoidal structure. Denote the symmetric monoidal unit in $\mathcal{C}$ by $\mathbf{1}$. Let $\mathcal{A} = \mathrm{Ab}$ be (the nerve of) the category of Abelian groups. For the functor $\pi: \mathcal{C} \to \mathcal{A}$ we take $[\mathbf{1}, -]$. For the category $\mathcal{I}$, we take a permutative skeleton of $\mathrm{Pic}(\mathcal{C})$. Recall that such a skeleton always exists by [JO12, Theorem B]. Denote the symmetric monoidal unit in $\mathcal{I}$ by 0. If X is a commutative algebra object in $\mathcal{C}$, then $\pi_\star(X)$ is a commutative algebra object in $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Ab})$. The commutative algebra object $\pi_\star(X)$ has the following graded-commutative multiplication: Let $V, W \in \mathcal{I}$, and let $x \in \pi_V(X)$, and $y \in \pi_W(X)$. Then

$$xy = uyx$$

where u is a certain unit in $\pi_0(X)$. The details of this are worked out in Equation (3.6). For now, the unit u arises as follows: Since $\mathcal{I}$ is skeletal, the objects $V \otimes W$ and $W \otimes W$ are identical. So the braiding $\tau: V \otimes W \to W \otimes V$ determines an element in $\mathrm{Aut}_{\mathcal{I}}(V \otimes W)$. Since every object in $\mathcal{I}$ is $\otimes$ invertible, τ determines an automorphism f of the unit $0 \in \mathcal{I}$. The element u is the image of $1 \in \pi_0(X)$ under $\pi_\star(f)$

Now we specailize Equation (2.9) to produce familiar examples.

Example 2.10. Consider Equation (2.9) in the case of $\mathcal{C} = \mathrm{Sp}$. In this setting, by Equation (1.7), we can write down a permutative skeleton of $\mathrm{Pic}(\mathrm{Sp}^G)$ explicitly. In this case $\pi_\star$ recovers the usual graded homotopy groups of a spectrum. If X is a homotopy ring spectrum, then $\pi_\star(X)$ is a graded-commutative ring in the usual sense.

Remark 2.11. There is a subtly. There are more objects in $\mathrm{Fun}(\mathcal{P}\mathrm{ic}(\mathrm{Sp})^{\mathrm{op}}, \mathrm{Ab})$ than there are what would usually be considered as a graded abelian group. The usual notion of a graded abelian group corresponds to things in the image of the graded homotopy groups functor. For concreteness, consider $\mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$. An object in $\mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$ consist of a collection of abelian groups $\{A_n\}_{n \in \mathbb{Z}}$, along with actions of $\mathrm{Aut}(\mathbb{Z})$ on A_n for each $n \in \mathbb{Z}$. Identify $\mathrm{Aut}(\mathbb{Z})$ with $\mathbb{Z}/2\mathbb{Z}$ and let τ be the non-identity element in $\mathrm{Aut}(\mathbb{Z})$. For a general object in $\mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$, it need not be the case that τ acts by -1. However, for any object that is in the image of $\pi_\star$, it is the case that $\tau \cdot x = -x$ for any $x \in A_n$.

Example 2.12. Consider Equation (2.9) in the case that $\mathcal{C} = \mathrm{Sp}^G$ for G any compact Lie group. This gives a good notion of graded homotopy groups for G-spectra. In particular, if X is a commutative algebra object in Sp^G, then $\pi_\star(X)$ has the following graded-commutative multiplication: If $x \in \pi_V(X)$, and $x \in \pi_W(X)$, then

$$xy = uyx$$

where $u \in \pi_0(X)$ is the unit determined by the braiding of V and W.

The following example, shown to me by Sven van Nigtevecht, shows that grading over $\mathrm{Pic}(\mathcal{C})$ sometimes simplifies the situation.

Example 2.13. Consider Equation (2.9) in the case $\mathcal{C} = \mathrm{Mod}_{\mathrm{KU}}$. The Picard group of KU is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. It follows that a permutative skeleton of $\mathrm{Pic}(\mathrm{Mod}_{\mathrm{KU}})$ contains only two objects. In this case $\pi_\star$ gives $\mathbb{Z}/2\mathbb{Z}$-graded homotopy groups.

Remark 2.14. Furthermore, we may take $\mathcal{I}$ to be any full subcategory of a permutative skeleton. This is useful, for example, in the case of equivariant spectra, where we may want to consider only the representation spheres in $\mathcal{P}\mathrm{ic}(\mathrm{Sp}^G)$, or, for example, in $K(n)$-local spectra, we may only want to consider the usual spheres rather than any exotic objects in $\mathcal{P}\mathrm{ic}(\mathrm{Sp}_{K(n)})$.

Example 2.15. Consider Equation (2.9) in the case that $\mathcal{C} = \mathrm{Syn}_E$ is the category of E-based synthetic spectra. Take $\mathcal{I}$ to be a full subcategory of a permutative skeleton of $\mathrm{Pic}(\mathrm{Syn}_E)$ generated by the bigraded spheres $\{\mathbb{S}^{s,t} := \Sigma^{-t}\nu\mathbb{S}^{s+t}\}$. Then $\pi_\star$ recovers the usual bigraded synthetic homotopy groups.

Example 2.16. Consider Equation (2.9) in the case that $\mathcal{C} = \mathrm{SH}(k)$ is the category of motivic spectra. Take $\mathcal{I}$ to be a full subcategory of a permutative skeleton of $\mathrm{Pic}(\mathrm{SH}(k))$ generated by the bigraded spheres $\{\mathbb{S}^{s,w} := \Sigma^{-w}\mathbb{G}_m^{\wedge(s+w)}\}$. Then $\pi_\star$ recovers the usual bigraded motivic homotopy groups.

In a different flavour, we may take $\mathcal{I} \to \mathrm{Pic}(\mathcal{C})$ to be any symmetric monoidal functor. This can be used to get a notion closer to the usual treatment of $RO(G)$-grading in terms of formal differences of isomorphism classes of representations.

Example 2.17. Consider the case of $\mathcal{C} = \mathrm{Sp}^G$ for G any compact Lie group. Let Lin^G be the topological category of finite dimensional G-equivariant inner product spaces with maps equivariant linear isometric isomorphisims. One can explicitly construct a symmetric monoidal functor of topological categories $\mathrm{Lin}^G \to \mathrm{Top}_*^G$ sending $V \mapsto S^V$. This induces a symmetric monoidal functor on the the corresponding ∞-categories which we denote $\mathcal{L}\mathrm{in}^G \to \mathcal{S}_*^G$. Post-composing with Σ^∞ gives a symmetric monoidal functor $\mathcal{L}\mathrm{in}^G \to \mathrm{Sp}^G$. Note, this functor factors over the inclusion $\mathcal{P}\mathrm{ic}(\mathrm{Sp}^G) \hookrightarrow \mathrm{Sp}^G$. So we have a symmetric monoidal functor $\mathcal{L}\mathrm{in}^G \to \mathcal{P}\mathrm{ic}(\mathrm{Sp}^G)$. Upon group completion, this gives a symmetric monoidal functor $(\mathcal{L}\mathrm{in}^G)^{\mathrm{gr}} \to \mathcal{P}\mathrm{ic}(\mathrm{Sp}^G)$. If we take $\mathcal{I} = (\mathcal{L}\mathrm{in}^G)^{\mathrm{gr}}$, this gives a good notion of '$RO(G)$'-graded homotopy groups. Taking group completion plays the same role as taking formal differences in the usual construction of '$RO(G)$'-grading [MC96, page 130].

3 Graded commutative algebra

In this section, we restrict to the case that $\mathcal{I}$ is a full subcategory of a permutative skeleton and $\mathcal{A} = \mathrm{Ab}$. The goal is to show that a good theory of commutative algebra can be built on $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Ab})$.

The advantage of grading over a groupoid becomes apparent when considering commutative algebra objects. Typically, one defines a graded ring as a collection of abelian groups with collections of maps that together define a notion of a graded multiplication. Working with this collection of maps, and actually checking that the collection of maps assemble into a well defined graded multiplication is tedious. However, since working with gradings over a groupoid gives a good category of graded abelian groups, it is now easy to define a graded ring as a commutative algebra object in graded abelian groups.

3.1 Graded abelian groups

First we setup some notation for dealing with $\mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Ab})$. For brevity, we leave the $\mathcal{I}$ implicit.

Definition 3.1. We define the category of *graded abelian groups* as the category

$$\mathrm{grAb} := \mathrm{Fun}(\mathcal{I}^{\mathrm{op}}, \mathrm{Ab})$$

equipped with Day convolution symmetric monoidal structure. For any object $A \in \mathrm{grAb}$, we denote evaluation at $V \in \mathcal{I}$ by A_V. For any morphism, $\varphi: V \to W$ we note its image under A by $\varphi^*: A_W \to A_W$.

Remark 3.2. The following shows that the notation φ^* is suitable: Suppose $A = \pi_\star(X)$ for some $X \in \mathcal{C}$. Let $x \in A_W$. Then x is represented by a map $W \xrightarrow{f} X$. For any $\varphi: V \to W$, the element $\varphi^* x$ is represented by $V \xrightarrow{\varphi} W \xrightarrow{f} X$.

We can give an explicit description of an object in grAb.

Remark 3.3. An object in grAb consists of a collection of abelian groups A_V, along with group homomorphisms

$$\mathrm{Hom}_{\mathcal{I}}(V, W) \to \mathrm{Hom}_{\mathrm{Ab}}(A_W, A_V).$$

Since $\mathcal{I}$ is skeletal, $\mathrm{Hom}_{\mathcal{I}}(V, W) = 0$ unless $V = W$. So the above group homomorphisms can be encoded as group actions

$$\mathrm{Aut}_{\mathcal{I}}(V) \times A_V \to A_V.$$

Furthermore, since every object in $\mathcal{I}$ is invertible, the symmetric monoidal structure induces isomorphisms $\mathrm{Aut}_{\mathcal{I}}(0) \to \mathrm{Aut}_{\mathcal{I}}(V)$ where 0 is the symmetric monoidal unit in $\mathcal{I}$. So, the above data is equivalent to a collection of abelian groups A_V, each with an action of $\mathrm{Aut}_{\mathcal{I}}(0)$.

3.2 Commutative algebra objects

In this section, we consider commutative algebra objects in grAb. The main goal of this section is to work out the details in Equation (2.9) describing commutative algebra objects in terms of graded abelian groups with a certain graded-commutative multiplication.

Definition 3.4. We denote by CAlg the category of commutative algebra objects in grAb.

Remark 3.5. Since the symmetric monoidal structure on grAb is given by Day convolution, a commutative algebra object in grAb is a lax symmetric monoidal functor

$$A: \mathcal{I}^{\mathrm{op}} \to \mathrm{Ab}.$$

A lax symmetric monoidal functor comes with the data of a unit map $\epsilon: \mathbf{1} \to A_0$, and the data of multiplication maps

$$\mu_{V,W}: A_V \otimes A_W \to A_{V \otimes W}.$$

The multiplication maps are required to be natural in V and W. The usual unitality, associativity, and symmetry diagrams are required to commute.

Fix a commutative algebra object A. For any $V, W \in \mathcal{I}$, the following diagram is required to commute.

$$\begin{tikzcd} A_V \otimes A_W \arrow[r] \arrow[d, "\mu_{V,W}"'] & A_W \otimes A_V \arrow[d, "\mu_{W,V}"] \\ A_{V\otimes W} \arrow[r] & A_{W\otimes W} \end{tikzcd}$$

The top horizontal map is the braiding in Ab of A_V and A_W. The bottom horizontal map is the A applied to the braiding $\tau_{V,W}\colon V \otimes W \to W \otimes V$ in $\mathcal{I}$. For $x \in A_V$, $y \in A_W$ this gives

$$\tau^*_{V,W} xy = yx.$$

Since $\mathcal{I}$ is skeletal, the objects $V \otimes W$ and $W \otimes V$ are identical. In particular, $\tau_{V,W}$ is an element in $\mathrm{Aut}_{\mathcal{I}}(V \otimes W)$. Since every object in $\mathcal{I}$ is invertible, the symmetric monoidal structure induces an isomorphism $\mathrm{Aut}_{\mathcal{I}}(0) \to \mathrm{Aut}_{\mathcal{I}}(V \otimes W)$. Under these isomorphisms, any braiding $\tau_{V,W}$ defines an element $g_{V,W} \in \mathrm{Aut}_{\mathcal{I}}(0)$. Using the action

$$\mathrm{Aut}_{\mathcal{I}}(0) \times A_0 \to A_0$$

any such $g_{V,W}$ defines an element $g_{V,W} \cdot 1 = u_{V,W} \in A_0$. By the definition of $u_{V,W}$ we have

$$u_{V,W} xy = yx$$

for $x \in A_V$, $y \in A_W$. Thus we have shown the following:

Lemma 3.6. *Let $A \in \mathrm{CAlg}$. For each $V, W \in \mathcal{I}$, there is an element $u_{V,W} \in A_0$ such that for any $x \in A_V$ and $y \in A_W$*

$$u_{V,W} xy = yx.$$

Remark 3.7. Since $\mathcal{I}$ is a permutative skeleton, $\tau_{V,W}\tau_{V,W} = \mathrm{id}_{V\otimes W}$. It follows that $u^{-1}_{V,W} = u_{W,V}$. In particular, the elements $u_{V,W}$ are units.

Notation 3.8. We refer to the collection of elements $u_{V,W}$ for all $V, W \in I$ as the *units that encode the graded-commutativity* in A.

3.3 Modules

Fix a commutative algebra object A. As one expects, we can consider left and right modules over A. We write LMod_A and RMod_A for the categories of left and right A-modules respectively.

A useful class of examples of A-modules are given by shifts of A. For any $V \in \mathcal{I}$ we define a functor

$$\Sigma^V\colon \mathrm{grAb} \to \mathrm{grAb}$$

by precomposing with $V \otimes -$.

Lemma 3.9. *Let $A, B \in \mathrm{grAb}$. Let $V, W \in \mathcal{I}$. There is an equivalence*

$$f_{V,W} \colon \Sigma^V A \otimes \Sigma^W B \to \Sigma^{V \otimes W}(A \otimes B)$$

that is natural in V and W.

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{I} \times \mathcal{I} \arrow[r, "V\otimes - \times W \otimes -"] \arrow[d, "\otimes"'] & \mathcal{I} \times \mathcal{I} \arrow[r, "A \times B"] \arrow[d, "\otimes"] & \mathrm{Ab} \times \mathrm{Ab} \arrow[r, "\otimes_{\mathrm{Ab}}"] & \mathrm{Ab} \\
\mathcal{I} \arrow[r, "(V \otimes W)\otimes -"'] & \mathcal{I} & &
\end{tikzcd}$$

Note, the left square only commutes up to a natural isomorphism. By definition, $A \otimes B$ is the left Kan extension of $\otimes_{\mathrm{Ab}} \circ A \times B$ along the tensor product of $\mathcal{I}$. By definition, $\Sigma^V A \otimes \Sigma^W B$ is the left Kan extension of $\otimes_{\mathrm{Ab}} \circ \Sigma^V A \times \Sigma^W B$ along the tensor product of $\mathcal{I}$. The left square commutes up-to natural isomorphism. Moreover, the horizontal morphisms in the left square are equivalences. So $\Sigma^V A \otimes \Sigma^W B$ is naturally isomorphic to the precomposing $A \otimes B$ with $(V \otimes W) \otimes -$. This is precisely $\Sigma^{V \otimes W} A \otimes B$. □

Lemma 3.10. *For any $V \in \mathcal{I}$ there is an equivalence $\Sigma^V A \cong \Sigma^V \mathbf{1} \otimes A$. Therefore, if M is a right A-module, then so is $\Sigma^V M$. Furthermore, for any $x \in A_V$, left multiplication by x defines a map of right A-modules*

$$l_x \colon M \to \Sigma^V M.$$

In particular, if A is a commutative algebra object $\Sigma^V A$ is a right A-module.

Proof. The fact that $\Sigma^V A \cong \Sigma^V \mathbf{1} \otimes A$ follows from Equation (3.9). The rest of the lemma follows by standard arguments. □

Remark 3.11. If Σ^V were instead defined by precomposition with $- \otimes V$, then $\Sigma^V A$ would be a left A-module. Similarly, right multiplication would define a map of left A-modules.

4 Localization

To demonstrate the convenience and manageability of grading over groupoids, we establish well-behaved localizations for objects in CAlg. Since our graded rings are commutative algebra objects in a nice category, the construction is mostly formal. We follow the more streamlined approach used by Hesselholt-Pstrągowski in [HP23].

Let's outline the approach. Fix a commutative algebra object $A \in \mathrm{grAb}$. We continue to assume that $\mathcal{I}$ is a full subcategory of a permutative skeleton of $\mathcal{P}\mathrm{ic}(\mathcal{C})$, and that $\mathcal{A} = \mathrm{Ab}$.[5] Since $\mathcal{I}$ is skeletal, we may define a *multiplicatively closed subset* $S \subset A$ to be a collection of homogenous elements that is closed under the multiplication in A. To any multiplicatively closed subset S we associate a category S/S. This category encodes how elements of S are multiplied in A. The role of S/S is to be a suitable diagram to take a colimit over. The localization $S^{-1}A$ will be defined as a colimit over S/S. To have a good handle on the colimit, we want S/S to be a filtered category. In the setting of [HP23], this is automatic. However, when working with more general gradings, mild conditions are needed to ensure that S/S is filtered.

Remark 4.1. Localizations can always be constructed abstractly. However, such abstract constructions can be unwieldy and may not behave as expected. By following [HP23], and constructing localizations in a more manual way, the resulting localization are more well behaved. For example, we would like the following to hold for $S^{-1}A$:

- Every element in $S^{-1}A$ can be represented as a fraction a/s that can be manipulated as expected.
- The kernel of the universal map $A \to S^{-1}A$ consists of precisely those elements in A such that $sa = 0$ for some $s \in S$.

If $S^{-1}A$ is constructed abstractly using a universal property, it need not be the case that either of these desired properties hold.

The construction of the category S/S is exactly as in [HP23].
Construction 4.2. Let S be a multiplicatively closed subset. We define a category S/S as having objects the elements of S, and morphisms $t\colon s_1 \to s_2$ whenever $ts_1 = s_2$.

The conditions for S/S being filtered are conveniently expressed as follows:

(F1) For any $s_1, s_2 \in S$, there are $t_1, t_2 \in S$ so that $t_1 s_1 = t_2 s_2$.

(F2) For any $s, s_1, s_2 \in S$ such that $s_1 s = s_2 s$, there exists $t \in S$ such that $ts_1 = ts_2$.

In [HP23], these are automatic. However in the current setting, we need additional conditions for (F2) to hold. However, these conditions are mild. For example, it is sufficient to assume S contains all of the units that encode the graded-commutativity conditions.

Remark 4.3. If S is of the form $\{1, f, f^2, \ldots\}$, then (F2) always holds. In a similar flavour, one can define, as one would expect, *prime ideals* of A. If S is the compliment of a prime ideal then (F2) always holds.

[5]The construction in this section does work more generally than the case of $\mathcal{A} = \mathrm{Ab}$, but it is not always the correct approach. For example, for Mackey functors, the correct approach requires a parameterized version of this setup.

Lemma 4.4. *If S contains the elements $u_{V,W}$ from Equation (3.6) for all $V, W \in \mathcal{I}$, then the category S/S is filtered.*

Proof. For (F1), we may take $t_1 = s_2$ and $t_2 = us_1$ where u is the unit encoding $s_2 s_2 = us_1 s_2$. For (F2), we make take $t = u_{V,W} s$ where $V = \deg(s)$ and $W = \deg(s_1) = \deg(s_2)$. □

From now on, we assume that S contains all the units encoding the graded-commutativity conditions. That is, for all $V, W \in \mathcal{I}$, S contains the elements $u_{V,W}$ constructed in Equation (3.6).

Remark 4.5. Since the elements $u_{V,W}$ are units, including them in S does not change the localization.

Now we construct the functor whose colimit will be used to define $S^{-1}A$.

Construction 4.6. We define a functor

$$\mathrm{Frac}\colon S/S \to \mathrm{RMod}_A,$$

on objects by $s \mapsto \Sigma^{\deg(s)} A$, and on morphisms by

$$(t\colon s_1 \to s_2) \mapsto \left(l_t\colon \Sigma^{\deg(s_1)} A \to \Sigma^{\deg(s_2)} \right).$$

Using Equation (3.10), it is straightforward to check that this is a well defined.

Now we define the localization.

Definition 4.7. We define $\gamma\colon A \to S^{-1}A$ as the universal map from A to the colimit of the functor Frac from Equation (4.6). In particular, we define $S^{-1}A$ as the colimit of Frac.

Theorem 4.8. *The localization $\gamma\colon A \to S^{-1}A$ satisfies the following:*

(1) Every element in $S^{-1}A$ can be represented as a fraction a/s with $a \in A$ and $s \in S$. The map γ is given by $a \mapsto a/1$. These fractions can be manipulated as expected: That is, $a_1/s_1 = a_2/s_2$ if there are $t_1, t_2 \in S$ such that $t_1 s_1 = t_2 s_2$ and $t_1 a_1 = t_2 a_2$. The right A-module structure is given by of $b/s \cdot a = ba/s$.

(2) The kernel of $\gamma\colon A \to S^{-1}A$ consists precisely of those elements of A such that $sa = 0$ for some $s \in S$.

(3) Every element of S is sent to a unit under γ.

Proof. Clearly both (2) and (3) follow from (1). We show (1). Colimits in a functor category are computed pointwise. Filtered colimits in Ab are computed in the category of sets. So we may use the description of a filtered colimit in sets. Write (a, s) to denote an element $a \in \mathrm{Frac}(s)$. Write a/s to denote the equivalence class of (a, s) in the equivalence relation given by the description of a filtered colimit. Spelling out the definition of Frac, and the definition of morphisms in S/S, we learn than $a_1/s_1 = a_2/s_2$ if there are $t_1, t_2 \in S$ such that $t_1 s_1 = t_2 s_2$ and $t_1 a_1 = t_2 a_2$. That the right A-module structure is given by of $b/s \cdot a = ba/s$ follows form computing the action on a representative (b, s). The fact γ is given by $a \mapsto a/1$ follows from the fact that $\mathrm{Frac}(1) = A$. □

Remark 4.9. The localization $S^{-1}A$ can be given a commutative algebra structure by

$$a/s \cdot b/t = abu_{V,W}/ts$$

where $V = \deg(a)$ and $W = \deg(t)^{-1}$. There are two ways to see this: One can work by hand and show that the same proof as in [Lam09, Theorem 10.6] works in this context The key idea is to treat a/s as $s^{-1}a$. Alternatively, one can show that the functor

$$\mathrm{Frac}\colon S/S \to \mathrm{RMod}_A,$$

is lax symmetric monoidal; hence the colimit is a commutative algebra in RMod_A.

Finally, we see that the localization satisfies the usual universal property.

Theorem 4.10. *The localization $\gamma\colon A \to S^{-1}A$ is initial among maps of commutative algebra objects that send every element of S to a unit. That is, if $f\colon A \to B$ is a map of commutative algebra objects where every element of S is sent to a unit, then there is a unique map of commutative algebra objects g making the following commute*

$$\begin{tikzcd} A \arrow[r, "\gamma"] \arrow[dr, "f"'] & S^{-1}A \arrow[d, dotted, "\exists! g"] \\ & B. \end{tikzcd}$$

Proof. The unique map g is given by $g(a/s) = (f(s))^{-1} f(a)$. The details are the same as in [Lam09, Corollarly 10.11]. □

A Universal property of Day convolution

Here we record some known facts about Day convolution. First we recall Lurie's construction of Day convolution. Fix an ∞-operad $\mathcal{O}^{\otimes}$. Let $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a co-Cartesian fibration of ∞-operads. Let $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be any fibration of ∞-operads. In [Lur17, Construction 2.2.6.7], Lurie defines $\mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes}$ as a norm of $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\times}$ along p. By the definition of a norm [Lur17, Definition 2.2.6.1], this implies that for any map ∞-operads $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ there is an equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}\left(\mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})\right) \to \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{C}}\left(\mathcal{C} \times_{\mathcal{O}} \mathcal{D}\right). \tag{A.1}$$

In this section we spell out this equivalence in the case that $\mathcal{O} = \mathrm{Fin}_*$, and the map of ∞-operads $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ is a symmetric monoidal ∞-category $\mathcal{E} \to \mathrm{Fin}_*$.

Lemma A.2. *The data of a lax symmetric monoidal functor $\mathcal{E} \times \mathcal{C} \to \mathcal{D}$ is equivalent to the data of a lax symmetric monoidal functor $\mathcal{E} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.*

Remark A.3. This is certainly known. It is implicit in [Gla16]. Jaco Ruit showed me this arguement.

Proof. First we look at the left side of A.1. In the case that $\mathcal{O} = \mathrm{Fin}_*$, and $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ is a symmetric monoidal category $\mathcal{E} \to \mathrm{Fin}_*$, the left hand side is

$$\mathrm{Alg}_{\mathcal{E}/\,\mathrm{Fin}_*}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))\,.$$

By [Lur17, Remark 2.1.3.2], this is equivalent to

$$\mathrm{Alg}_{\mathcal{E}}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))\,.$$

By [Lur17, Example 2.2.6.9], objects in here are precisely lax symmetric monoidal functors $\mathcal{E} \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Now consider the right side of A.1. By definition we have

$$\mathrm{Alg}_{\mathcal{E}\times_{\mathcal{O}}\mathcal{C}/\mathcal{C}}(\mathcal{C} \times_{\mathcal{O}} \mathcal{D}) = \mathrm{Map}_{/\mathcal{C}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes}).$$

This mapping space sits in the following coCartesian square

$$\begin{tikzcd}
\mathrm{Map}_{/\mathcal{C}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Map}_{/\mathcal{O}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes}) \arrow[d] \\
* \arrow[r] & \mathrm{Map}_{/\mathcal{O}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{C}^{\otimes}),
\end{tikzcd}$$

where the bottom map picks out the map $\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}$. The right vertical arrow sits in the following coCartesian square

$$\begin{tikzcd}
\mathrm{Map}_{/\mathcal{O}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Map}_{/\mathcal{O}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \arrow[d] \\
\mathrm{Map}_{/\mathcal{O}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{C}^{\otimes}) \arrow[r] & *.
\end{tikzcd}$$

Pasting these squares gives a coCartesian square

$$\begin{tikzcd}
\mathrm{Map}_{/\mathcal{C}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes}) \arrow[r] \arrow[d] & \mathrm{Map}_{/\mathcal{O}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \arrow[d] \\
* \arrow[r] & *.
\end{tikzcd}$$

Since the bottom map is an equivalence, so is the top map. Hence

$$\mathrm{Alg}_{\mathcal{E}\times_{\mathcal{O}}\mathcal{C}/\mathcal{C}}(\mathcal{C} \times_{\mathcal{O}} \mathcal{D}) \cong \mathrm{Map}_{/\mathcal{O}^{\otimes}}(\mathcal{E}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}).$$

Now, using that $\mathcal{O} = \mathrm{Fin}_*$, an object in this mapping space is precisely a lax symmetric monoidal functor $\mathcal{E} \times \mathcal{C} \to \mathcal{D}$. $\square$

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