Multiplicative Equivariant Thom Spectra & Structured Real Orientations

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Abstract

For strongly even $\mathbb{E}_\infty^{C_2}$-rings E we show that any homotopy ring map $\mathrm{MU} \to E^e$ lifts to an $\mathbb{E}_\rho$-map $\mathrm{MU}_\mathbb{R} \to E$. This refines the Hahn–Shi Real orientations of Lubin–Tate theories E_n, the Hirzebruch level-n orientations of $\mathrm{tmf}_1(n)$, and Quillen's idempotent to $\mathbb{E}_\rho$-maps. This allows us to provide the first structured version of $\mathrm{BP}_\mathbb{R}$ – we show that it admits an $\mathbb{E}_\rho$-algebra structure. Furthermore, we extend these results to larger groups. In particular, for a finite group $C_2 \leq G$ the Hahn–Shi orientation $N_{C_2}^G \mathrm{MU}_\mathbb{R} \to E_n$ refines to a $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-map, and $N_{C_2}^G \mathrm{BP}_\mathbb{R}$ admits a $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-algebra structure.

Essential to this program is a robust theory of multiplicative equivariant Thom spectra, which we develop using parametrized higher algebra and fibrous patterns – particularly we provide an equivariant version of Antolín-Camarena–Barthel's universal property for multiplicative Thom spectra and use this to deduce a multiplicative equivariant Thom isomorphism. We provide a number of categorical results of independent interest, most notably a distributive monoidal structure on parametrized left module categories.

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Summary of Results

We develop obstruction-theoretic methods to produce structured orientations. As a consequence, we are able to give a first answer to the open question about multiplicative structures on the Real Brown–Peterson spectrum and its norms. Let ρ denote the regular representation of C_2.

Theorem (Theorem 7.3.1)**.** Let $C_2 \leq G$. There is a $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-algebra structure on $N_{C_2}^G \mathrm{BP}_{\mathbb{R}}$.

Prior to this work, not even the existence of an $\mathbb{E}_1$- or $\mathbb{E}_\sigma$-algebra structure was known on $\mathrm{BP}_{\mathbb{R}}$. One reason for the sparsity of results in this direction is the lack of computationally tractable equivariant obstruction theories. We obtain this structure on $\mathrm{BP}_{\mathbb{R}}$ and its norms by developing a general obstruction theory for producing structured orientations.

Theorem (Corollary 7.1.5, Theorem 7.2.1)**.** Let E be a strongly even $\mathbb{E}_\infty^{C_2}$-ring spectrum.

(i) Then, E admits an $\mathbb{E}_\rho$-$\mathrm{MU}_{\mathbb{R}}$-orientation.

(ii) Any $\mathbb{E}_2$-ring map $\mathrm{MU} \to E^e$ lifts uniquely to an $\mathbb{E}_\rho$-ring map $\mathrm{MU}_{\mathbb{R}} \to E$. In particular, any homotopy ring map $\mathrm{MU} \to E^e$ lifts uniquely to an $\mathbb{E}_\rho$-ring map.

This immediately refines several orientations of interest, answering a question in [AKKQ25, Remark 5.7].

Example (Corollary 7.2.3, Corollary 7.2.4, Corollary 7.2.6)**.** Let $C_2 \leq G$.

(i) The Hahn–Shi Real orientations $N_{C_2}^G \mathrm{MU}_{\mathbb{R}} \to E_n$ admit lifts to $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-maps.

(ii) The Hirzebruch level-n genera $\mathrm{MU}_{\mathbb{R}} \to \mathrm{tmf}_1(n)$ admit lifts to $\mathbb{E}_\rho$-maps.

(iii) The Burklund–Hahn–Levy–Schlank Adams operations ψ^ℓ on $\mathrm{MU}_{(2)}$ admit lifts to $\mathbb{E}_\rho$-maps $\mathrm{MU}_{\mathbb{R}(2)} \to \mathrm{MU}_{\mathbb{R}(2)}$.

Our main tool is a multiplicative version of the equivariant Thom isomorphism.

Theorem (Theorem 4.2.9)**.** Let G be a finite group and V be a G-representation. Suppose that $R \to A$ is a map of $\mathbb{E}_\infty^G$-ring spectra and that $f\colon X \to \underline{\mathrm{Pic}}_G(R)$ is a map of $\mathbb{E}_V$-monoidal spaces. An $\mathbb{E}_V$-A-orientation of f gives rise to a Thom isomorphism

$$A \otimes_R \mathrm{Th}_G(f) \simeq A \otimes \Sigma^\infty_+ X$$

of $\mathbb{E}_V$-A-algebras.

We prove several results in parametrized higher algebra to obtain the above theorem. Most notably, we need good monoidal structures on parametrized left module categories.

Theorem (Theorem 3.3.7, Theorem 3.3.14)**.** Let $\underline{\mathscr{C}}^\otimes$ be a G-symmetric monoidal G-∞-category and $\underline{\mathscr{O}}^\otimes$ be a G-∞-operad, both satisfying certain conditions. Let $A \in \mathbf{Alg}_{\mathscr{O} \otimes \mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathscr{C}})$.

(i) The pullback

$$\begin{tikzcd}
\underline{\mathbf{LMod}}_A^G(\underline{\mathscr{C}})^\otimes \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathcal{LM}}(\underline{\mathscr{C}})^\otimes \arrow[d] \\
\underline{\mathscr{O}}^\otimes \arrow[r, "A"'] & \underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathscr{C}})^\otimes
\end{tikzcd}$$

is an $\underline{\mathscr{O}}$-monoidal G-∞-category.

(ii) If $\underline{\mathscr{C}}^\otimes$ is $\underline{\mathscr{O}}$-distributive, then so is $\underline{\mathbf{LMod}}_A^G(\underline{\mathscr{C}})^\otimes$.

Contents

1 Introduction

The $K(n)$-local sphere $L_{K(n)}\mathbb{S}$ is one of the fundamental objects of study in chromatic homotopy theory. By Devinatz–Hopkins [DH04], there is an equivalence $L_{K(n)}\mathbb{S} \simeq E_n^{h\mathbb{G}}$, where E_n is a height n Lubin–Tate theory, and $\mathbb{G}$ is the extended Morava stabilizer group. Hopkins–Miller [Rez98] observed that, by taking finite subgroups $G \leq \mathbb{G}$, one can produce higher height versions of KO, i.e. in the same way that KO detects substantial v_1-periodic information, the spectra

$$\mathrm{EO}_n(G) = E_n^{hG}$$

detect substantial v_n-periodic information. For example, for $n = 1$, this is closely related to KO, and for $n = 2$ this is closely related to TMF. Thus, the *higher real K-theories* EO_n contain rich information about the stable homotopy groups of spheres.

Hill–Hopkins–Ravenel's landmark work [HHR09] began as an attempt to compute $E_4^{hC_8}$. They showed that $E_4^{hC_8}$ would detect the Kervaire classes [HHR10]. However, the action on E_n is obtained through the Hopkins–Miller Theorem, rendering the associated homotopy fixed point spectral sequence intractable. Nonetheless, there was good evidence this action could be made tractable. At height 1, the Real Conner–Floyd orientation $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KU}_{\mathbb{R}}$ relates the formal-group-theoretic action to the geometric complex conjugation action. In celebrated work, Hahn–Shi [HS20] produced higher height analogues of the Real Conner-Floyd orientation, namely the *Hahn–Shi Real orientations* $\mathrm{MU}_{\mathbb{R}} \to E_n$ (and normed versions thereof). They thereby provided the first computation of $E_n^{hC_2}$ for $n > 2$, and opened the door to applying the Hill–Hopkins–Ravenel program to understanding higher real K-theories [BBHS20, BHSZ21, HLS21, HSWX23, DLS25, CH25, DHL+25b].

However, from a multiplicative perspective, Real orientations are still poorly understood. As pointed out in [AKKQ25, Remark 5.7], it was previously not known whether E_n or $\mathrm{tmf}_1(n)$ admit $\mathbb{E}_\rho$-$\mathrm{MU}_{\mathbb{R}}$-orientations. Such orientations are, among others, essential for applications in Real trace methods: an $\mathbb{E}_\sigma$-algebra structure is required to define $\mathrm{THR}(-)$, and an $\mathbb{E}_\rho$-$\mathrm{MU}_{\mathbb{R}}$-orientation allows one to define $\mathrm{THR}(-/\mathrm{MU}_{\mathbb{R}})$. One of the main goals of this article is to construct structured orientations of the aforementioned examples.

The sparsity of multiplicative results in the Real setting stems from a lack of computationally tractable equivariant obstruction theories. Even from a theoretical point of view, most work thus far has focused on the fully commutative case via $\mathbb{N}_\infty$-operads. This excludes many of the most important ring spectra from consideration: Moore spectra $\mathbb{S}/p^k$, Morava K-theories $K(n)$, Ravenel's $X(n)$ spectra, and the Brown–Peterson spectrum BP all do not admit $\mathbb{E}_\infty$-ring structures [Rog08, Law18, ACB19, Dev24, Sen24]. Consequently, equivariant multiplicative refinements of these are inaccessible through $\mathbb{N}_\infty$-operads.

We are led to study equivariant versions of $\mathbb{E}_n$, and to develop tools to answer the following guiding problem:

Problem. For which C_2-representations V does $\mathrm{BP}_{\mathbb{R}}$ admit an $\mathbb{E}_V$-algebra structure?

Partial Answer. It admits an $\mathbb{E}_\rho$-algebra structure (Theorem 7.3.1). □

This question has received much attention in the non-equivariant setting [BM13, CM15, Law18, HW22, Sen24, DHL+25a], leading to many advances about structured ring spectra. Nonetheless, the Real question remained completely open.

Equivariant little disk operads $\mathbb{E}_V$ – like the non-equivariant ones – come in a hierarchy controlling the amount of equivariant commutativity of their associated algebras. In particular, they specify norms and associated coherence maps according to the given representation. The specific structure of such algebras was studied by Hill [Hil22b]. As an example, consider the C_2-sign representation σ. Then, an $\mathbb{E}_\sigma$-algebra $A^\otimes$ can be viewed as a C_2-spectrum $A \in \mathbf{Sp}_{C_2}$ with underlying $\mathbb{E}_1$-algebra $\mathrm{Res}_e^{C_2} A$ and a left module structure $N_e^{C_2} \mathrm{Res}_e^{C_2} A \otimes A \to A$ [Hor19, Section 7.1]. We will use this module structure to discuss nilpotence results for $\mathbb{E}_\sigma$-algebras (Section 8.3).

Attacking the question about structures on $\mathrm{BP}_\mathbb{R}$ leads us to structured versions of *equivariant Thom spectra*. In ∞-categorical language, Ando–Blumberg–Gepner–Hopkins–Rezk [ABG+14] give an elegant formulation of Thom spectra: Let R be an $\mathbb{E}_\infty$-ring spectrum, X be a space and $f\colon X \to \mathrm{Pic}(R)$ be a map of spaces. Then, its Thom spectrum is defined as an ∞-categorical colimit

$$\mathrm{Th}(f) = \mathrm{M}f = \operatorname{colim}\left(X \xrightarrow{f} \mathrm{Pic}(R) \to \mathbf{LMod}_R\right).$$

This can be made equivariant via the recent surge of *parametrized higher category* theory [BDG+16, CLL23, Hil24b, MW24, Ste25a]. All the defining notions admit parametrized analogues, so we can define equivariant Thom spectra as G-colimits in the exact same fashion. This motivates using the toolkit of parametrized higher category theory to transport results about non-equivariant Thom spectra to equivariant ones.

This ∞-categorical formalism allows a more accessible treatment of multiplicative structures on Thom spectra, although they were already classically studied by Lewis [LMSM86]. Concurrently, Ando–Blumberg–Gepner [ABG18] and Antolín-Camarena–Barthel [ACB19] analyzed the compatibility of Th with multiplicative objects. Notably, Antolín-Camarena–Barthel proved a universal property of multiplicative Thom spectra through operadic left Kan extensions and used this to obtain a multiplicative Thom isomorphism.

Let $f\colon X \to \underline{\mathrm{Pic}}_G(R)$ be a map of G-spaces and A be an R-algebra. Its Thom spectrum is informally a twisted version of $R \otimes X$, and a detwist after a base change is called orientation. More precisely, an *A-orientation* of a map $f\colon X \to \underline{\mathrm{Pic}}_G(R)$ takes one of the following equivalent forms (Corollary 4.2.5):

(i) A lift

$$\begin{tikzcd} & \mathrm{GL}_1(\underline{\mathrm{Pic}}_G(R)_{\downarrow A}) \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "f"'] & \underline{\mathrm{Pic}}_G(R) \end{tikzcd}$$

of f.

(ii) A nullhomotopy of the composite

$$X \xrightarrow{f} \underline{\mathrm{Pic}}_G(R) \xrightarrow{\mathrm{Ind}_R^A} \underline{\mathrm{Pic}}_G(A).$$

in $\mathcal{S}_G$.

(iii) An R-module map $\mathrm{Th}_G(f) \to A$ such that for every $x\colon * \to X$ the adjoint A-module map corresponding to the R-module map

$$\mathrm{Th}_G(f \circ x) \longrightarrow \mathrm{Th}_G(f) \longrightarrow A$$

is an equivalence.

The importance of orientations lies in the resulting Thom isomorphisms $A \otimes_R \mathrm{Th}_G(f) \simeq A \otimes X$. Each of the above notions admit structured refinements, called *structured orientations*, inducing structured versions of the Thom isomorphism. An interest in structured versions of the Thom isomorphism already started long ago, see Mahowald–Ray [MR81].

A part of this article consists of mimicking the work of Antolín-Camarena–Barthel [ACB19] via parametrized higher category theory to lift their results to the equivariant setting. This will provide basic tools to develop an obstruction theory for structured equivariant orientations.

1.1 Main Results & Outline

Foundations

The main goal of our foundational work is to develop a robust theory of multiplicative equivariant Thom spectra that allows us to discuss orientation theory (Section 6), and an accompanying multiplicative Thom isomorphism (Theorem 4.2.9). Since orientations are about base changes between ring spectra, a good theory of modules over rings is essential. This motivates our work on an equivariant left module category **LMod**, and its monoidal properties (Section 3.3). In the subsequent discussion we will be a little vague in giving the precise assumptions since those are often quite technical to state. The precise statements can be found in the body of the article.

There is a notion of inflated G-∞-operads (Construction 2.2.1), that turns an ∞-operad $\mathscr{O}^{\otimes}$ into a G-∞-operad $\mathrm{Infl}_G\, \mathscr{O}^{\otimes}$. Algebras for $\mathrm{Infl}_G\, \mathscr{O}^{\otimes}$ are simply levelwise $\mathscr{O}$-algebras. So given a G-symmetric monoidal G-∞-category $\underline{\mathscr{C}}^{\otimes}$ and an $\mathrm{Infl}_G\, \mathbb{E}_1$-algebra A therein, one can mimic the classical setting to define

$$\begin{tikzcd} \underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathbf{Alg}}_{\mathrm{Infl}_G\, \mathcal{LM}}(\underline{\mathscr{C}}) \arrow[d] \\ * \arrow[r, "A"'] & \underline{\mathbf{Alg}}_{\mathrm{Infl}_G\, \mathbb{E}_1}(\underline{\mathscr{C}}) \end{tikzcd}$$

where $\mathcal{LM}$ is the classical left-module operad [Lur17, Section 4.2.1]. Stewart extended the Boardman–Vogt tensor product to G-∞-operads (Theorem 2.2.4) and equipped the above algebra categories with G-symmetric monoidal structures. So given a G-∞-operad $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$, a map $\mathscr{O}^{\otimes} \to \underline{\mathbf{Alg}}_{\mathrm{Infl}_G\, \mathbb{E}_1}(\underline{\mathscr{C}})^{\otimes}$ of G-∞-operads corresponds to an $\mathscr{O}^{\otimes} \otimes \mathrm{Infl}_G\, \mathbb{E}_1$-algebra in $\underline{\mathscr{C}}^{\otimes}$.

We use this observation to study monoidal structures on **LMod**. The core difficulty lies in proving that **LMod** is distributive in the sense of Nardin–Shah (Definition 2.3.8), which says that the G-monoidal structure is compatible with G-colimits in a suitable sense.

Theorem A (Theorem 3.3.7, Theorem 3.3.14)**.** Let $\underline{\mathscr{C}}^{\otimes}$ be a G-symmetric monoidal G-∞-category and $\underline{\mathscr{O}}^{\otimes}$ be a G-∞-operad, both satisfying certain conditions. Let $A \in \mathbf{Alg}_{\mathscr{O} \otimes \mathrm{Infl}_G\, \mathbb{E}_1}(\underline{\mathscr{C}})$.

(i) The pullback

$$\begin{tikzcd} \underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathbf{Alg}}_{\mathrm{Infl}_G\, \mathcal{LM}}(\underline{\mathscr{C}})^{\otimes} \arrow[d] \\ \underline{\mathscr{O}}^{\otimes} \arrow[r, "A"'] & \underline{\mathbf{Alg}}_{\mathrm{Infl}_G\, \mathbb{E}_1}(\underline{\mathscr{C}})^{\otimes} \end{tikzcd}$$

is an $\mathscr{O}$-monoidal G-∞-category.

(ii) If $\underline{\mathscr{C}}^{\otimes}$ is $\underline{\mathscr{O}}$-distributive, then so is $\underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}})^{\otimes}$.

Proof Sketch.

(i) First, this pullback can be taken in $\mathbf{Op}_{G,\infty}$, so $\underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}})^\otimes \to \underline{\mathscr{O}}^\otimes$ is a map of G-∞-operads. It remains to show that the left vertical arrow is a coCartesian fibration. Since coCartesian fibrations pull back, it suffices to show that the right vertical arrow is a coCartesian fibration. This is classical [Lur17, Lemma 4.5.3.6] for each equivariant level

$$\underline{\mathbf{Alg}}_{\mathcal{LM}}(\underline{\mathscr{C}})^\otimes_H \to \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\underline{\mathscr{C}})^\otimes_H$$

and to pass to the G-symmetric monoidal algebra categories we use a criterion by Haugseng–Melani–Safronov (Corollary 3.3.6) for which we essentially need to check that norms, fiberwise tensor products and restrictions must preserve coCartesian edges.

(ii) We equivariantize the following strategy, which henceforth becomes notationally much heavier. Consider the two colimit diagrams $I^\triangleright, J^\triangleright \to \mathbf{LMod}_A(\mathscr{C})$ and the following diagram.

$$\begin{tikzcd}
(I\times J)^\triangleright \arrow[r] \arrow[d, equal] & I^\triangleright \times J^\triangleright \arrow[r] \arrow[d, equal] & \mathbf{LMod}_A(\mathscr{C})^{\times 2} \arrow[r, "\otimes"] \arrow[d] & \mathbf{LMod}_{A\otimes A}(\mathscr{C}) \arrow[r, "A\otimes_{A\otimes A}-"] \arrow[d] & \mathbf{LMod}_A(\mathscr{C}) \\
(I\times J)^\triangleright \arrow[r] & I^\triangleright \times J^\triangleright \arrow[r] & \mathscr{C}^{\times 2} \arrow[r, "\otimes"'] & \mathscr{C} &
\end{tikzcd}$$

The last two functors of the top composite compose to the tensor product in $\mathbf{LMod}_A(\mathscr{C})$ and we verify such a factorization for our equivariant setting (Proposition 3.3.13). The main part of distributivity is to show that the top line is a colimit diagram. Distributivity of $\mathscr{C}^\otimes$ shows that the bottom line is a colimit diagram. Since the vertical forgetful functor reflects colimits, also the composite of the top three functors is a colimit diagram. But then, $A \otimes_{A\otimes A} -$ preserves this since it is a left adjoint. □

For us, this was the most difficult ingredient to get many tools from parametrized higher algebra running. Distributivity is required to use operadic left Kan extensions (Theorem 2.3.13). After setting up a suitable notion of G-Picard spaces (Construction 3.4.16), along with their G-monoidal structures, we can construct multiplicative equivariant Thom spectra. Start with a map $f\colon X^\otimes \to \underline{\mathrm{Pic}}^\otimes_G(A)$ between $\underline{\mathscr{P}}$-monoidal G-spaces for some (suitable) G-∞-operad $\underline{\mathscr{P}}^\otimes$.[1] The operadic version of left Kan extensions then allows us to define

$$\begin{tikzcd}
X^\otimes \arrow[r, "f"] \arrow[d] & \underline{\mathrm{Pic}}^\otimes_G(A) \arrow[r] & \underline{\mathbf{LMod}}^G_A(\underline{\mathbf{Sp}}_G)^\otimes \\
\underline{\mathscr{P}}^\otimes \arrow[urr, dashed, bend right, "\mathrm{Th}^\otimes_G(f)"'] & &
\end{tikzcd}$$

which is a multiplicative enhancement of G-Thom spectra. In particular, this shows that a $\underline{\mathscr{P}}$-monoidal map $f\colon X^\otimes \to \underline{\mathrm{Pic}}^\otimes_G(A)$ gives rise to a $\underline{\mathscr{P}}$-algebra refinement of $\mathrm{Th}_G(f)$. This extension is essentially defined by a universal property, an equivariant version of Antolín-Camarena–Barthel's universal property. In particular, this makes producing maps of out $\mathrm{Th}^\otimes_G(f)$ tractable.

Theorem B (Theorem 4.1.8)**.** The functor

$$\mathrm{Th}^\otimes_G\colon \mathbf{Alg}_{\underline{\mathscr{P}}}(\underline{\mathcal{S}}_G)_{/\underline{\mathrm{Pic}}^\otimes_G(A)} \to \mathbf{Alg}_{\underline{\mathscr{P}}}\left(\underline{\mathbf{LMod}}^G_A(\underline{\mathbf{Sp}}_G)\right)$$

admits an explicitly described right adjoint in terms of G-Picard spaces.

[1] We write $(-)^\otimes$ to indicate that the monoidal structure is part of the data.

If $\mathrm{Th}_G^{\otimes}(f)$ is an R-module Thom spectrum with $\underline{\mathscr{P}}$-algebra structure, then a $\underline{\mathscr{P}}$-A-orientation is a $\underline{\mathscr{P}}$-map $\mathrm{Th}_G^{\otimes}(f) \to A$ such that on all fibers the adjoint A-module map is an equivalence (Corollary 4.2.5). This generalizes the classical Thom class notion and such maps can be understood through the aforementioned universal property (Theorem B). So it does not require much more to massage this into the multiplicative Thom isomorphism.

Theorem C (Theorem 4.2.9)**.** A $\underline{\mathscr{P}}$-A-orientation of the map f gives rise to an equivalence $A \otimes_R \mathrm{Th}_G^{\otimes}(f) \simeq A \otimes X^{\otimes}$ of $\underline{\mathscr{P}}$-A-algebras.

During the course of this categorical endeavour, we needed to prove a number of additional results, which we expect to be of independent interest. As such, we provide a monoidal structure on parametrized slice categories (Section 3.1) through certain cotensors, put together results from the literature to give a weak version of parametrized monoidal straightening-unstraightening (Corollary 3.2.3) and discuss monoidal equivariant versions of the GL_1-functor, which universally takes the units of a multiplicative G-space (Corollary 3.4.12).

Remark 1.1.1. For the sake of readability, we don't work in the highest possible generality.

(i) Throughout the entire article, G will always be a finite group. Our main reason for this is that the theory of parametrized higher algebra is not yet suitably worked out for non-finite groups. Once this changes, our results should go through mutatis mutandis.

(ii) We always parametrize over $\mathbf{Orb}_G$ instead of a general parametrizing base ∞-category $\mathcal{T}$. If one is careful enough with the correct adjectives, like atomic orbital, existence of terminal objects, and so on, then our results also work in this higher generality.[2]

(iii) We expect that the Part I results are also true in a global equivariant context for normed global ring spectra by replacing the span pattern $\mathrm{Span}(\mathbb{F}_G)$ by a global analog.

The main difference lies in the distributivity of **LMod** since the Nardin–Shah setup of distributivity does not apply in the global setup. In this setting, there is a theory set up by Lenz–Linskens–Pützstück [LLP25] who use drastically different language. Nonetheless, the same distributivity proof idea should apply here.

The main work is already done by Lenz–Linskens–Pützstück who prove that $\underline{\mathbf{Sp}}_{\mathrm{gl}}^{\otimes}$ is distributive [LLP25, Theorem 5.10]. So this already provides the theory of global Thom spectra over the global sphere spectrum $\mathbb{S}_{\mathrm{gl}}$.

Applications

For applications we mostly focus on Real equivariant homotopy theory, but we formulate a number of results for more general finite groups G out of independent interest.

Let X be a strongly even C_2-spectrum (Definition 5.1.3). Let ρ be the regular representation of C_2. Strong evenness implies the restriction map $\mathrm{res}_e^{C_2} \colon \pi_{*\rho}^{C_2}(X) \to \pi_{2*}^{e}(X)$ is an isomorphism, whence C_2-equivariant information can be deduced from underlying non-equivariant information. We extend this notion to positively indexed towers of C_2-spectra demanding that the first term and all associated graded pieces are strongly even (Definition 5.1.3). By an induction and limiting argument, we obtain a similar conclusion that the restriction map is an isomorphism of the limit of towers (Proposition 5.2.4). Together with a Thom isomorphism computation, we deduce the following lifting result for structured orientations.

[2] In fact, this is how we wrote a preliminary draft before we decided to prioritize readability over generality.

Theorem D (Theorem 6.1.4)**.** Let R be an $\mathbb{E}_\infty^{C_2}$-ring spectrum and E be a strongly even $\mathbb{E}_\infty^{C_2}$-algebra in $\underline{\mathbf{LMod}}_R^{C_2}$. Suppose that X is an $n\rho$-loop space with a map $f : X \to \underline{\mathrm{Pic}}_G(R)$ of $n\rho$-loop spaces. Suppose that there exists an $\mathbb{E}_{n\rho}$-R-algebra map $\mathrm{M}f \to E$ and suppose that X satisfies certain finiteness and evenness conditions. Then, the restriction map

$$\mathrm{res}_e^{C_2} : \pi_{*\rho}^{C_2}\left(\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})}(\mathrm{M}f, E)\right) \to \pi_{2*}^{e}\left(\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})}(\mathrm{M}f, E)\right)$$

is an isomorphism.

Proof Sketch. After detwisting the left side of $\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})}(\mathrm{M}f, E)$ by the Thom isomorphism (Theorem C), we can massage this term with a sequence of formal reformulations (Proposition 6.1.1) giving rise to

$$\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{G})}(\mathrm{M}f, E) \simeq \Omega^\infty \underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty \mathrm{B}^{n\rho} X, \Sigma^{n\rho}\mathrm{gl}_1(E))$$

where $\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}$ denotes the C_2-mapping spectrum. Via a cohomological slice tower, we can reduce this to the strong even property as alluded to above. □

This is our main abstract lifting result, which we will mainly apply to the Real bordism spectrum $\mathrm{MU}_\mathbb{R}$ to study structured Real orientations. In particular, we upgrade Chadwick–Mandell's result about $\mathbb{E}_2$-orientations [CM15, Theorem 1.2] to the Real setting.

Theorem E (Corollary 7.1.5, Theorem 7.2.1)**.** Let E be a strongly even $\mathbb{E}_\infty^{C_2}$-ring spectrum.

(i) Then, E admits an $\mathbb{E}_\rho$-$\mathrm{MU}_\mathbb{R}$-orientation.

(ii) Any $\mathbb{E}_2$-ring map $\mathrm{MU} \to E^e$ lifts uniquely to an $\mathbb{E}_\rho$-ring map $\mathrm{MU}_\mathbb{R} \to E$. In particular, any homotopy ring map $\mathrm{MU} \to E^e$ lifts uniquely to an $\mathbb{E}_\rho$-ring map.

Proof Sketch. Proof ingredients of (i) are used again in (ii) which is why we phrase the theorem this way.

(i) Let $\overline{\gamma}$: $\mathrm{MU}_\mathbb{R} \to \mathrm{ku}_\mathbb{R}$ be the Conner–Floyd Real orientation of $\mathrm{ku}_\mathbb{R}$. Then, we consider the C_2-Thom spectrum

$$\mathrm{MW}_\mathbb{R} = \mathrm{Th}_{C_2}\left(\Omega^\infty\Sigma^\rho\,\mathrm{MU}_\mathbb{R} \to \Omega^\infty\Sigma^\rho\,\mathrm{ku}_\mathbb{R} \simeq \mathrm{BU}_\mathbb{R} \to \underline{\mathrm{Pic}}_{C_2}(\underline{\mathbf{Sp}}^{C_2})\right),$$

which is a C_2-Thom spectrum over a Real Wilson space. Angelini-Knoll–Kong–Quigley realize that $\mathrm{MW}_\mathbb{R}$ has a nice $\mathbb{E}_\infty^{C_2}$-ρ-cellular decomposition [AKKQ25, Theorem 5.10], so that by obstruction theory there exists an $\mathbb{E}_\infty^{C_2}$-map $\mathrm{MW}_\mathbb{R} \to E$ (see Theorem 7.1.3).

Inspired by a decomposition of Hahn–Raksit–Wilson [HRW24, Theorem 3.2.10] of MW into copies of MU, we split $\mathrm{MU}_\mathbb{R}$ off of $\mathrm{MW}_\mathbb{R}$ as an $\mathbb{E}_\rho$-retract, thus providing an $\mathbb{E}_\rho$-map $\mathrm{MU}_\mathbb{R} \to \mathrm{MW}_\mathbb{R}$.

(ii) Given an $\mathbb{E}_2$-ring map $\mathrm{MU} \to E^e$, we may precompose it with the map $\mathrm{MW} \to \mathrm{MU}$ from (i). Results of Hill–Hopkins on the homology of Real Wilson spaces ensure the finiteness and evenness conditions of Theorem D, so we may use it obtain an $\mathbb{E}_\rho$-lift $\mathrm{MW}_\mathbb{R} \to E$ of $\mathrm{MW} \to E^e$. Precomposing this with $\mathrm{MU}_\mathbb{R} \to \mathrm{MW}_\mathbb{R}$ from the splitting in (i) then gives an $\mathbb{E}_\rho$-map $\mathrm{MU}_\mathbb{R} \to E$ lifting the given $\mathbb{E}_2$-ring map $\mathrm{MU} \to E^e$.

The statement about lifting homotopy ring maps $\mathrm{MU} \to E^e$ now follows since any such map uniquely lifts to an $\mathbb{E}_2$-map by Chadwick–Mandell [CM15, Theorem 1.2]. □

Moreover, this also provides a Real version of the Chadwick–Mandell result [CM15, Theorem 1.2] for $\mathbb{E}_\infty^{C_2}$-rings, as any Real orientation $\mathrm{MU}_\mathbb{R} \to E$ forgets down to a homotopy ring map $\mathrm{MU} \to E^e$ which then lifts to an $\mathbb{E}_\rho$-map $\mathrm{MU}_\mathbb{R} \to E$.

Our theory can then be immediately applied to equip familiar examples with more structure.

Example F (Corollary 7.2.3, Corollary 7.2.6)**.**

(i) The Hahn–Shi Real orientations [HS20] of Lubin–Tate theories $\mathrm{MU}_\mathbb{R} \to E_n$ admit lifts to $\mathbb{E}_\rho$-maps.

(ii) The Hirzebruch level-n genera [Mei23] $\mathrm{MU}_\mathbb{R} \to \mathrm{tmf}_1(n)$ admit lifts to $\mathbb{E}_\rho$-maps.

(iii) The Burklund–Hahn–Levy–Schlank [BHLS23] Adams operations on $\mathrm{MU}_{(2)}$ admit lifts to $\mathbb{E}_\rho$-operations on $\mathrm{MU}_{\mathbb{R}(2)}$.

Let $H \leq G$ and $\mathscr{O}^\otimes$ be an H-∞-operad. If A is an $\mathscr{O}$-ring spectrum, then $N_H^G A$ always obtains the structure of an $\mathrm{Coind}_H^G \mathscr{O}$-algebra by parametrized abstract nonsense (Construction 2.2.5). This allows us to enhance all our C_2-level results to larger groups (Corollary 7.1.7).

Example G (Corollary 7.2.4)**.** Let $C_2 \leq G$.

(i) The Hahn–Shi Real orientations $\mathrm{MU}^{((G))} = N_{C_2}^G \mathrm{MU}_\mathbb{R} \to E_n$ refine to $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-maps.

(ii) Let $n > 1$ and $G = (\mathbb{Z}/n)^\times$. Then, the Hirzebruch level-n genera induce $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-maps $\mathrm{MU}^{((G))} \to \mathrm{tmf}_1(n)$.

Finally, we use our results to provide a first structured version of the Real Brown–Peterson spectrum $\mathrm{BP}_\mathbb{R}$. Classically, BP is constructed through Quillen's idempotent $\mathrm{MU}_{(2)} \to \mathrm{MU}_{(2)}$ [Qui69]. By our lifting results (Theorem E), this provides an $\mathbb{E}_\rho$-refinement of the Real Quillen idempotent $\mathrm{MU}_{\mathbb{R}(2)} \to \mathrm{MU}_{\mathbb{R}(2)}$ [Ara79]. After checking that this again defines an idempotent (Theorem 6.2.1), it splits off an $\mathbb{E}_\rho$-version of $\mathrm{BP}_\mathbb{R}$.

Theorem H (Theorem 7.3.1)**.** Let $C_2 \leq G$. Then, there is an $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-algebra structure on $\mathrm{BP}^{((G))} = N_{C_2}^G \mathrm{BP}_\mathbb{R}$.

1.2 Relation to Other Work

This article touches many areas that have been discussed by other authors, all of whom we have benefited from.

(i) *Multiplicative equivariant Thom spectra.* These were considered by Horev–Klang–Zou [HHK+24]. We fill in some details and in particular avoid a strong version of parametrized monoidal straightening-unstraightening (Assumption 8.1.1), which is stated as folklore in their article [HHK+24, Theorem A.6.1]. We take the further step of proving a universal property of multiplicative equivariant Thom spectra generalizing the Antolín-Camarena–Barthel universal property [ACB19]. Furthermore, we consider left module spectra over more base rings than just the sphere spectrum $\mathbb{S}$, which takes a considerable amount of additional work.

(ii) *Equivariant monoidal module categories.* Monoidal parametrized versions of module categories have been considered by other authors like Linskens–Nardin–Pol, Yang, and Pützstück [LNP25, Yan25b, Yan25a, Pü25]. Moreover, Blumberg–Hill studied equivariant module categories through models [BH20]. We make the effort of working over all finite groups, incorporating (almost) all G-operads and prove the hard fact of distributivity (Theorem 3.3.14).

(iii) *Structured orientations.* Our Theorem E is similar to a conjectured result by Roytman [Roy23, Anticipated Theorem] who outlined an approach and towards which he made partial progress. We take a different approach. Roytman's approach is to directly generalize the tools used in Chadwick–Mandell's paper [CM15] to the equivariant setting, and then directly generalize Chadwick–Mandell's argument. That is, Roytman's strategy is to show that Real orientations can be lifted to $\mathbb{E}_\rho$-maps. This approach requires equivariant versions of TAQ which have not appeared in the literature.

We instead prove that $\mathbb{E}_2$-maps $\mathrm{MU} \to E^e$ lift to $\mathbb{E}_\rho$-maps $\mathrm{MU}_\mathbb{R} \to E$, which ultimately also shows that Real orientations can be lifted to $\mathbb{E}_\rho$-maps by first applying Chadwick–Mandell's original theorem [CM15, Theorem 1.2] before applying our result. We take this approach for three main reasons:

- it generalizes to more than $\mathbb{E}_\rho$-maps, we get obstruction theory for lifting $\mathbb{E}_{2n}$-maps to $\mathbb{E}_{n\rho}$-maps,
- it generalizes to arbitrary equivariant R-module Thom spectra, in particular it does not rely on the close connection between Real orientations and homotopy commutative ring maps,
- all of the technical machinery required follows once we generalize the work of Antolín-Camarena–Barthel [ACB19] to the equivariant setting.

On the other hand, Roytman's conjecture differs from our Theorem E by relaxing the requirement of an $\mathbb{E}_\infty^{C_2}$-structure to an $\mathbb{E}_\rho$-structure. Given the equivariant TAQ needed for Roytman's strategy, the results of our paper can also be sharpened.

1.3 Overview

We split our article into two parts: Part I concerns the categorical framework of multiplicative equivariant Thom spectra via parametrized higher algebra; Part II applies this to obtain concrete results, mostly about structured Real orientations. The reader who is predominately interested in the applications – and willing to believe that equivariant multiplicative Thom spectra work 'as expected' – is free to skip directly to Part II. For the convenience of the reader, we sketch a proof for the $\mathbb{E}_\rho$-structure on $\mathrm{BP}_\mathbb{R}$ at the beginning of Part II that accommodates those who skip Part I. It demonstrates the main ideas of Part II while avoiding certain technicalities that arise when studying general orientations.

In Section 2 we recall the main notions from parametrized higher algebra and set up notation. In particular, we define singly-coloured, inflated, and coinduced G-∞-operads, and give an extensive overview of Nardin–Shah distributivity.

In Section 3 we begin by providing a monoidal structure on parametrized slice ∞-categories and proceed by assembling results from the literature to give a (weak) monoidal version of parametrized straightening-unstraightening. Then, we do the hard work of setting up a robust monoidal version of equivariant left module categories and check Nardin–Shah distributivity. We end with a discussion of equivariant grouplike algebras in which we recall equivariant versions of the recognition theorem and work out suitable versions of GL_1 and Pic.

In Section 4 we discuss parametrized multiplicative Thom spectra and prove an equivariant version of the Antolín-Camarena–Barthel universal property of multiplicative Thom spectra. We then set up (multiplicative) orientation theory through a universal example and notably prove a multiplicative equivariant Thom isomorphism.

In Section 5 we review and study strongly even spectra, which we extend to towers. Furthermore we set up and study a cohomological version of the equivariant slice tower.

In Section 6 we set up our main obstruction theory to study structured orientations of equivariant Thom spectra by reducing the problem to a Bredon cohomology computation via the multiplicative Thom isomorphism. We then specialize to self-orientations and study idempotents and their splittings in this regard.

In Section 7 we specialize to the Real setting for our main applications. We begin by producing $\mathbb{E}_\rho$-orientations for strongly even $\mathbb{E}_\infty^{C_2}$-rings through Real Wilson spaces. Such orientations allow us to use the Thom isomorphism to produce preferred lifts of orientations. In particular, we show that underlying complex orientations $\mathrm{MU} \to E^e$ of strongly even $\mathbb{E}_\infty^{C_2}$-ring spectra lift to $\mathbb{E}_\rho$-Real orientations $\mathrm{MU}_\mathbb{R} \to E$. We apply this to equip examples from the literature with more structure: the Hahn–Shi orientations, the Hirzebruch level-n genera, and the Adams operations on MU. Furthermore, we use this to refine Quillen's idempotent whence we construct an $\mathbb{E}_\rho$-algebra structure on $\mathrm{BP}_\mathbb{R}$. For $C_2 \leq G$ we enhance each of those to $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$.

In Section 8 we give additional applications. We give a formula for relative THR of equivariant Thom spectra, rephrase equivariant versions of the Hopkins–Mahowald theorem in terms of $\mathbb{E}_V$-quotients, and end with a curiosity related to the Hopkins-Mahowald theorem in the context of nilpotence, namely that $\mathrm{MU}_\mathbb{R}$ detects nilpotence for $\mathbb{E}_\sigma$-rings and that $\mathrm{H}\underline{\mathbb{Z}}$ detects nilpotence for $\mathbb{E}_\sigma \otimes \mathbb{E}_\infty$-rings.

In Section A we spell out that restrictions and norms of G-symmetric monoidal G-∞-categories are symmetric monoidal functors. We conclude by explaining the existence of an $\mathbb{E}_\infty^{C_2}$-structured Real J-homomorphism $J_\mathbb{R} \colon \mathrm{BU}_\mathbb{R} \to \mathrm{Pic}_{C_2}(\underline{\mathbf{Sp}}_{C_2})$ via forthcoming work of Brink–Lenz.

1.4 Notations & Conventions

We collect notational conventions here that we will use throughout the whole article.

(1) Throughout, G will always be a finite group.[3]

(2) We denote by ρ the regular representation of C_2. More generally, ρ_G is the regular representation of G. Moreover, we write σ for the 1-dimensional C_2-sign representation.

(3) Underlined categories are G-∞-categories, e.g. $\underline{\mathbf{Sp}}_G$ is the G-∞-category which on level $H \leq G$ is $\mathbf{Sp}_H$.

(4) Let $X, Y \in \mathbf{Sp}_G$. We denote by $\underline{\mathrm{map}}_G(X, Y)$ the mapping G-spectrum refining the mapping G-space $\underline{\mathrm{Map}}_G(X, Y)$. We also write $\underline{\mathrm{map}}_{\mathbf{Sp}_G}(X, Y)$ and $\underline{\mathrm{Map}}_{\mathbf{Sp}_G}(X, Y)$.

(5) The G-∞-operad $\mathbb{E}_\infty^G$ is the terminal G-∞-operad. So $\mathbb{E}_\infty^G$-algebras admit all norms – they are also known as normed algebras, ultracommutative algebras, or G-$\mathbb{E}_\infty$-algebras.

(6) Let $H \leq G$ be a subgroup. In any parametrized/equivariant setup, we denote by Res_H^G the restriction functor and by Ind_H^G resp. Coind_H^G the left resp. right adjoint – this equivalently also appears as indexed coproducts resp. products via parametrized colimits resp. limits.

See also the Index of Notation.

[3]The finiteness can be negotiable (Remark 1.1.1).

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Part I

Foundations

Section 2 This section recalls the main notions from parametrized higher algebra and sets up the notation.

In Section 2.1 we recall the notion of G-∞-operads in the language of fibrous patterns due to Barkan–Haugseng–Steinebrunner but also mention underlying Nardin–Shah parametrized ∞-operads. We define singly colored G-∞-operads.

In Section 2.2 we recall Stewart's construction of inflated G-∞-operads and the algebras that they corepresent. Moreover, we discuss coinduced operads and the way they yield the natural structure on norms of algebras.

In Section 2.3 we give an extensive overview of Nardin–Shah distributivity in parametrized higher algebra. We spell out a model-independent viewpoint of parametrized fibers via parametrized straightening-unstraightening, spell out a projection formula as a consequence of distributivity, and recall operadic left Kan extensions.

Section 3 This is the main technical section of the article in which we assemble and prove all the categorical tools we will need.

In Section 3.1 we describe a monoidal structure on parametrized slice ∞-categories in the large generality of fibrous patterns. We give a universal property of this construction.

In Section 3.2 we assemble results from the literature to give a (weak) monoidal version of parametrized straightening-unstraightening.

In Section 3.3 we define parametrized left module categories and endow them with a suitably monoidal structure. The proof of this relies on an equivariant version of a result by Haugseng–Melani–Safronov to check that a functor of $\mathscr{O}$-monoidal ∞-categories is a coCartesian fibration. We then do the hard work of checking that $\mathbf{LMod}_A(-)$ preserves Nardin–Shah distributivity.

In Section 3.4 we first give an overview of equivariant grouplike algebras, notably equivariant versions of the recognition theorems by Cnossen–Haugseng–Lenz–Linskens, Guillou–May, and Juran; in particular, we spell out the interplay of the delooping functor with taking infinite loop spaces Ω^∞. We then work out an equivariantly multiplicative version of the GL_1 functor and finally define equivariant Picard spaces together with a monoidal structure on them. We end by discussing a comma category $\mathrm{Pic}_G^{\otimes}(R)_{\downarrow A}$ involving Picard spaces and describe $\mathrm{GL}_1(\mathrm{Pic}_G^{\otimes}(R)_{\downarrow A})$.

Section 4 This section combines the categorical work to discuss parametrized Thom spectra and the orientation theory that they control.

In Section 4.1 we recall the definition of parametrized Thom spectra and go on by giving several possibilities of multiplicatively enhancing this construction – namely via Day convolution and via operadic left Kan extension. We use the operadic left Kan extension viewpoint to give an equivariant version of the Antolín-Camarena–Barthel universal property of multiplicative Thom spectra.

In Section 4.2 we set up (multiplicative) orientation theory through a universal example. We then prove some results about multiplicative orientations, most notably a multiplicative Thom isomorphism.

2 Recollections on Parametrized Higher Algebra

2.1 Language of Parametrized Operads

Our article will build on the language of parametrized higher algebra [NS22] and fibrous patterns [BHS24]. Parametrized higher category theory works for (potentially nice enough) base ∞-categories $\mathcal{T}$, but we will restrict to the G-orbit category $\mathbf{Orb}_G$ for a finite group G for the sake of readability. We will begin by recalling some vocabulary while reminding the reader that the Barkan–Haugseng–Steinebrunner fibrous pattern language is supposed to be a (vast) generalization of ∞-operads. Please feel free to skip this and return to it to look up vocabulary.

(1) Recall that the *G-orbit category* $\mathbf{Orb}_G$ is the full subcategory of the ∞-category of G-spaces $\mathcal{S}_G$ spanned by the transitive G-sets $\{G/H\}_{H\leq G}$. A G-∞-category is a functor $\mathbf{Orb}_G^{\mathrm{op}} \to \mathbf{Cat}_\infty$, or equivalently a coCartesian fibration over $\mathbf{Orb}_G^{\mathrm{op}}$ by straightening-unstraightening: $\mathbf{Cat}_{G,\infty} = \mathrm{Fun}(\mathbf{Orb}_G^{\mathrm{op}}, \mathbf{Cat}_\infty) \simeq \mathrm{coCart}(\mathbf{Orb}_G^{\mathrm{op}})$.

(2) An *algebraic pattern* is an ∞-category $\mathscr{O}$ equipped with a factorization system of so-called *inert* and *active* morphisms and a collection of *elementary objects* [CH21]. For $o \in \mathscr{O}$ we write $\mathscr{O}^{\mathrm{el}}_{o/} = \mathscr{O}^{\mathrm{el}} \times_{\mathscr{O}^{\mathrm{int}}} \mathscr{O}^{\mathrm{int}}_{o/}$ where the superscripts stand for elementary objects resp. inert edges. We write $\mathbf{AlgPatt}$ for the ∞-category of algebraic patterns.

(3) Let $\mathscr{O} \in \mathbf{AlgPatt}$ and $\mathscr{C} \in \mathbf{Cat}_\infty$. A functor $F : \mathscr{O} \to \mathscr{C}$ is a *Segal $\mathscr{O}$-object* in $\mathscr{C}$ if for every $o \in \mathscr{O}$ the induced functor

$$(\mathscr{O}^{\mathrm{el}}_{o/})^{\triangleleft} \longrightarrow \mathscr{O} \xrightarrow{F} \mathscr{C}$$

is a limit diagram. We write $\mathbf{Seg}_{\mathscr{O}}(\mathscr{C})$ for the ∞-category of Segal $\mathscr{O}$-objects in $\mathscr{C}$ [CH21].

(4) Let $\mathscr{O} \in \mathbf{AlgPatt}$. A *fibrous $\mathscr{O}$-pattern* is a functor $\mathscr{P} \to \mathscr{O}$ admitting all coCartesian lifts of inert edges satisfying another condition akin to the classical ∞-operad conditions [BHS24, Definition 4.1.2]. We write $\mathbf{Fbrs}(\mathscr{O})$ for the ∞-category of fibrous $\mathscr{O}$-patterns.

(5) Let $\mathscr{O} \in \mathbf{AlgPatt}$ and $\mathscr{P}, \mathscr{Q} \in \mathbf{Fbrs}(\mathscr{O})$. Then, we write $\mathbf{Alg}_{\mathscr{P}/\mathscr{O}}(\mathscr{Q}) = \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O})}(\mathscr{P}, \mathscr{Q})$. We write $\mathbf{Alg}_{\mathscr{P}}(\mathscr{Q}) = \mathbf{Alg}_{\mathscr{P}/\mathscr{O}}(\mathscr{Q})$.

(6) There is an algebraic pattern structure on $\mathrm{Span}(\mathbb{F}_G)$, which is an example of a span pattern. Inert resp. actives are backwards resp. forward maps and the elementary objects are those objects in $\mathbf{Orb}_G$ [BHS24, Definition 3.2.6]. This will be the most relevant algebraic pattern for us.

(7) We denote by $\mathbf{Op}_{G,\infty} = \mathbf{Fbrs}(\mathrm{Span}(\mathbb{F}_G))$ the ∞-category of G-∞-operads.

(8) Let $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$, then we write $\mathbf{Mon}_{\mathscr{O}}(\mathbf{Cat}_\infty)$ for the subcategory of $\mathbf{Fbrs}(\mathscr{O}^{\otimes})$ spanned by the coCartesian fibrations with morphisms the ones preserving coCartesian edges. This is equivalent to $\mathbf{Seg}_{\mathscr{O}}(\mathbf{Cat}_\infty)$.

(9) Given an $\mathscr{O}$-monoidal ∞-category $\mathscr{C}^{\otimes} \to \mathscr{O}^{\otimes}$ and a map $o \to o'$, then we will denote by $\bigotimes_{o\to o'} : \mathscr{C}^{\otimes}_o \to \mathscr{C}^{\otimes}_{o'}$ the induced functor.

We will study many aspects of our paper in the language of patterns, which was not the original way parametrized higher algebra was phrased. Rather, Nardin–Shah [NS22] define G-∞-operads as coCartesian fibrations over a certain G-∞-category $\underline{\mathbb{F}}_{G,*}$ with certain operad conditions. On the other hand [BHS24, Proposition 5.2.14] shows that it is an equivalent notion, allowing us to pull back to the parametrized world along a map $\underline{\mathbb{F}}_{G,*} \to \mathrm{Span}(\mathbb{F}_G)$. Suitable comparison results can be found in the literature [BHS24, Pü24, Ste25a]. While we prefer to work with patterns, there are some parts where we cannot avoid working in Nardin–Shah's formalism without potentially expending much more effort in comparison results – most notably when discussing distributivity. In such cases, will decorate objects in the Nardin–Shah formalism with a superscript $(-)^{\mathrm{NS}}$, e.g. we will write $\mathbf{Op}_{G,\infty}^{\mathrm{NS}}$ and $\mathbf{Mon}_{\underline{\mathcal{O}}}^{\mathrm{NS}}(\underline{\mathbf{Cat}}_{G,\infty})$.

Before writing out terminologies of underlying Nardin–Shah objects, we first define singly G-colored G-∞-operads because we will need this to state the definition that follows.

Definition 2.1.1. A G-∞-operad $\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_G)$ has a single G-color/is singly G-colored if the projection map

$$\mathcal{O}^{\otimes} \times_{\mathrm{Span}(\mathbb{F}_G)} \mathbf{Orb}_G^{\mathrm{op}} \to \mathbf{Orb}_G^{\mathrm{op}}$$

is an equivalence.

In particular, we obtain a map $\mathbf{Orb}_G^{\mathrm{op}} \to \mathcal{O}^{\otimes}$ via the projection map. Moreover, there is a map $\underline{\mathbb{F}}_{G,*} \to \mathrm{Span}(\mathbb{F}_G)$.

Construction 2.1.2. Let $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$, $\mathcal{C}^{\otimes} \in \mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_{\infty})$ and $A^{\otimes} \in \mathbf{Alg}_{\mathcal{O}}(\mathcal{C})$.

(i) The underlying Nardin–Shah G-∞-operad of $\mathcal{O}^{\otimes}$ is $\underline{\mathcal{O}}^{\otimes} = \mathcal{O}^{\otimes} \times_{\mathrm{Span}(\mathbb{F}_G)} \underline{\mathbb{F}}_{G,*} \to \underline{\mathbb{F}}_{G,*}$.

(ii) The underlying $\underline{\mathcal{O}}$-monoidal G-∞-category of $\mathcal{C}^{\otimes}$ is $\underline{\mathcal{C}}^{\otimes} = \mathcal{C}^{\otimes} \times_{\mathrm{Span}(\mathbb{F}_G)} \underline{\mathcal{O}}^{\otimes} \to \underline{\mathcal{O}}^{\otimes}$ with the structure map $\underline{\mathcal{O}}^{\otimes} \to \underline{\mathbb{F}}_{G,*} \to \mathrm{Span}(\mathbb{F}_G)$ to define the fiber product on the left.

(iii) The underlying G-∞-category of $\mathcal{C}^{\otimes}$ is $\underline{\mathcal{C}} = \mathcal{C}^{\otimes} \times_{\mathrm{Span}(\mathbb{F}_G)} \mathbf{Orb}_G^{\mathrm{op}} \to \mathbf{Orb}_G^{\mathrm{op}}$.

(iv) Suppose now that $\mathcal{O}^{\otimes}$ has a single G-color and let $A^{\otimes} \colon \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$ be an $\mathcal{O}$-algebra. Its underlying G-object is given by

$$\underline{A} : \mathbf{Orb}_G^{\mathrm{op}} \simeq \mathcal{O}^{\otimes} \times_{\mathrm{Span}(\mathbb{F}_G)} \mathbf{Orb}_G^{\mathrm{op}} \to \mathcal{C}^{\otimes} \times_{\mathrm{Span}(\mathbb{F}_G)} \mathbf{Orb}_G^{\mathrm{op}} \simeq \underline{\mathcal{C}}.$$

Note for example that $\underline{\mathcal{C}} \to \mathbf{Orb}_G^{\mathrm{op}}$ is really a G-∞-category because $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ has all inert lifts and we are pulling back to a part of the inerts. By inspection, you can take underlying objects in various orders, e.g. $\underline{\mathcal{C}}$ is also the underlying G-∞-category of $\underline{\mathcal{C}}^{\otimes}$.

Moreover, we will also underline objects that already come from the parametrized higher algebra world.

Usually, the Nardin–Shah and Barkan–Haugseng–Steinebrunner formalisms will be interchangable and don't make a crucial difference in our arguments.

We end this subsection with a result on singly-colored G-∞-operads.

Lemma 2.1.3. Let $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ have a single G-color and $\mathcal{C}^{\otimes} \in \mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_{\infty})$. Then, its underlying G-∞-category can also be described as $\underline{\mathcal{C}} \simeq \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathbf{Orb}_G^{\mathrm{op}}$.

Proof. This is by pullback pasting applied to

$$\begin{tikzcd}
\underline{\mathcal{C}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}^{\otimes} \arrow[d] \\
\mathbf{Orb}_G^{\mathrm{op}} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & \mathcal{O}^{\otimes} \arrow[d] \\
\mathbf{Orb}_G^{\mathrm{op}} \arrow[r] & \mathrm{Span}(\mathbb{F}_G)
\end{tikzcd}$$

where the bottom square is a pullback square since $\mathcal{O}^{\otimes}$ has a single G-color and the composite square is a pullback by definition. Thus, we deduce that the top square is a pullback square by pullback pasting. $\square$

2.2 Inflated & Coinduced Operads

We already have many interesting operads $\mathscr{O}^{\otimes}$ from classical non-equivariant mathematics and wish to bump up these to inflated G-operads $\mathrm{Infl}_G\,\mathscr{O}^{\otimes}$. Algebras over $\mathrm{Infl}_G\,\mathscr{O}^{\otimes}$ should at each level be classical algebras over $\mathscr{O}^{\otimes}$. So with $\mathscr{O}^{\otimes} = \mathbb{E}_1^{\otimes}$ this will for example allow us to speak about a structure with a coherently associative multiplication on each level.

Construction 2.2.1. By functoriality of Span, the functor $i\colon \mathbb{F} \to \mathbb{F}_G$, $\underline{1} \mapsto G/G$ induces aother functor $i_*\colon \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}(\mathbb{F}_G)$. The associated pullback functor $i^*\colon \mathbf{Op}_{G,\infty} \to \mathbf{Op}_{\infty}$ then admits a left adjoint

$$\mathrm{Infl}_G = i_!\colon \mathbf{Op}_{\infty} \simeq \mathbf{Fbrs}(\mathrm{Span}(\mathbb{F})) \to \mathbf{Fbrs}(\mathrm{Span}(\mathbb{F}_G)) \simeq \mathbf{Op}_{G,\infty},$$

called inflation. For brevity, we will often omit the inflation functor from the notation. For example, there are the classical ∞-operads $\mathbb{E}_1^{\otimes}$ and $\mathcal{LM}^{\otimes}$, which we will view as G-∞-operads $\mathbb{E}_1^{\otimes} = \mathrm{Infl}_G\,\mathbb{E}_1^{\otimes}$ and $\mathcal{LM}^{\otimes} = \mathrm{Infl}_G\,\mathcal{LM}^{\otimes}$ by inflation.

Checks. Here are the technical fibrous pattern checks that we need to do to obtain the above functor.

(i) Stewart slightly generalizes [BHS24, Corollary 4.2.3] and gives a criterion for the existence of a left adjoint [Ste25a, Proposition 2.23]. We check those, i.e. we observe that $\mathrm{Span}(\mathbb{F}_G)$ is soundly extendable [Ste25a, Lemma A.8] and that i_* is a Segal morphism, which can be checked straight from the definition [Ste25a, Proposition A.19].

(ii) Stewart also has a construction of the inflation functor [Ste25a, below Proposition 3.22], which also has right adjoint i^*, so these inflation functors agree. □

Remark 2.2.2. Consider the functor $p\colon \mathbb{F}_G \to \mathbb{F}$, $G/H \mapsto \underline{1}$ for every $H \leq G$. This induces a functor $p^*\colon \mathbf{Op}_{\infty} \to \mathbf{Op}_{G,\infty}$ which is given by pulling back along $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Span}(\mathbb{F})$. This gives another way to construct G-∞-operads from ∞-operads but one does not obtain inflated operads. For example, the G-∞-operad $p^*\mathbb{E}_k^{\otimes}$ is what we believe should be called the *$\mathbb{E}_k^G$-operad*.

Question 2.2.3. Is there a good theory of $\mathbb{N}_k$-operads similar to that of $\mathbb{N}_\infty$-operads [BH15]? We believe that in such a theory $\mathbb{E}_k^{G,\otimes} = p^*\mathbb{E}_k^{\otimes}$ should be the one leading to the most norms on algebras, and hence should be called the $\mathbb{E}_k^G$-operad.

Theorem 2.2.4 ([Ste25a, Theorem D]). Let $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $\mathscr{C} \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_{\infty})$. There exists a functor

$$\otimes\colon \mathbf{Op}_{G,\infty} \times \mathbf{Op}_{G,\infty} \to \mathbf{Op}_{G,\infty}$$

refining the Boardman–Vogt tensor product such that:

(i) For $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ the functor $- \otimes \mathscr{O}^{\otimes}\colon \mathbf{Op}_{G,\infty} \to \mathbf{Op}_{G,\infty}$ admits a right adjoint $\mathbf{Alg}_{\mathscr{O}}(-)^{\otimes}$ refining the G-∞-category of $\mathscr{O}$-algebras.

(ii) We have $\mathbf{Alg}_{\mathscr{O}}(\mathscr{C})^{\otimes} \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_{\infty})$.

(iii) If $\mathscr{O}^{\otimes}$ has a single G-color, then the forgetful functor $\mathbf{Alg}_{\mathscr{O}}(\mathscr{C})^{\otimes} \to \mathscr{C}^{\otimes}$ is G-symmetric monoidal.

(iv) Let $\mathscr{O}^{\otimes} \in \mathbf{Op}_{\infty}$ and $H \leq G$. Then, $\mathbf{Alg}_{\mathrm{Infl}_G\,\mathscr{O}}(\mathscr{C})_H^{\otimes} \simeq \mathbf{Alg}_{\mathscr{O}}(\mathscr{C}_H)$ naturally.

Let $\mathscr{O}^{\otimes}, \mathscr{P}^{\otimes} \in \mathbf{Op}_{G,\infty}$. The adjunction in part (i) in particular shows

$$\mathbf{Alg}_{\mathscr{O}\otimes\mathscr{P}}(-)^{\otimes} \simeq \mathbf{Alg}_{\mathscr{O}}(\mathbf{Alg}_{\mathscr{P}}(-))^{\otimes}$$

by a Yoneda argument.

Let us end this subsection with a discussion on coinduced G-∞-operads. Its purpose is that for $H \le G$ with $\mathcal{O}^{\otimes} \in \mathbf{Op}_{H,\infty}$ and an $\mathcal{O}$-algebra A, its norm $N_H^G A$ naturally obtains an $\mathrm{Coind}_H^G \mathcal{O}$-algebra structure. The idea already appeared in [BH15, Section 6.2] which is phrased in parameterized language in [Ste25b].

Construction 2.2.5. Let $H \le G$.

(i) Postcomposing by the functor $\mathrm{Res}_H^G : \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Span}(\mathbb{F}_H)$ induces the restriction map $\mathrm{Res}_H^G : \mathbf{Op}_{G,\infty} \to \mathbf{Op}_{H,\infty}$. This admits a right adjoint $\mathrm{Coind}_H^G : \mathbf{Op}_{H,\infty} \to \mathbf{Op}_{G,\infty}$ by some parametrized higher category theory [Ste25b, Section 1.3.1]. It is the coinduction functor of equivariant operads.

(ii) Consider a singly H-colored $\mathcal{O}^{\otimes} \in \mathbf{Op}_{H,\infty}$ and the counit $\varepsilon : \mathrm{Res}_H^G \mathrm{Coind}_H^G \mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}$. Then, there is a commutative diagram of symmetric monoidal functors [Ste25b, Construction 1.40]

$$\begin{tikzcd}
\mathbf{Alg}_{\mathcal{O}}(\mathrm{Res}_H^G \mathscr{C}) \arrow[r, "\varepsilon^*"] \arrow[d] & \mathbf{Alg}_{\mathrm{Res}_H^G \mathrm{Coind}_H^G \mathcal{O}}(\mathrm{Res}_H^G \mathscr{C}) \arrow[r, "N_H^G"] \arrow[d] & \mathbf{Alg}_{\mathrm{Coind}_H^G \mathcal{O}}(\mathscr{C}) \arrow[d] \\
\mathscr{C}_H \arrow[r, equal] & \mathscr{C}_H \arrow[r, "N_H^G"'] & \mathscr{C}_G
\end{tikzcd}$$

The symmetric monoidality is by Lemma A.1.1 and the right square is via Theorem 2.2.4 (iii). To apply the latter, we need that $\mathrm{Coind}_H^G \mathcal{O}^{\otimes}$ is singly G-colored, which follows from the next result (Lemma 2.2.6).

Lemma 2.2.6. Let $H \le G$ and consider a singly H-colored $\mathcal{O}^{\otimes} \in \mathbf{Op}_{H,\infty}$. Then, the G-∞-operad $\mathrm{Coind}_H^G \mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ is singly G-colored.

Proof. By the arguments in [Ste25b, Section 1.3.1] there is a functor $\underline{\mathbf{Op}}_{G,\infty} \to \underline{\mathbf{Cat}}_{G,\infty}$ which on level $K \le G$ takes the underlying K-∞-category and strongly preserves parametrized limits. Thus, the diagram

$$\begin{tikzcd}
\mathbf{Op}_{H,\infty} \arrow[r, "\mathrm{Coind}_H^G"] \arrow[d] & \mathbf{Op}_{G,\infty} \arrow[d] \\
\mathbf{Cat}_{H,\infty} \arrow[r, "\mathrm{Coind}_H^G"'] & \mathbf{Cat}_{G,\infty}
\end{tikzcd}$$

commutes. The left arrow sends $\mathcal{O}^{\otimes}$ to the terminal object by the singly colored condition, so the composite sends to the terminal object because Coind_H^G is a right adjoint. In particular, $\mathrm{Coind}_H^G \mathcal{O}^{\otimes}$ is sent to $\mathbf{Orb}_G^{\mathrm{op}}$, i.e. is singly G-colored. □

2.3 Distributivity of Parametrized Monoidal Structures

One essential categorical tool in this article to realize multiplicative Thom spectra and their universal properties is through operadic left Kan extensions, which is roughly a lax monoidal version of extending by colimits. Such a notion should naturally require a compatibility of the monoidal structure with colimits. More specifically, the monoidal structure should distribute over colimits. The notion of distributivity was initiated by Nardin in his PhD thesis [Nar17] and further worked out by Nardin–Shah [NS22].

For the convenience of the reader, we will now give an exposition of this theory. On the way, we spell out the relation of parametrized fibers with parametrized straightening-unstraightening

(Proposition 2.3.4) and prove a projection formula assuming distributivity (Proposition 2.3.12). This is one of the sections where we are forced to work with the [NS22] formalism instead of the [BHS24] formalism because distributivity in this generality is only worked out there.

Remark 2.3.1. Recently, Lenz–Linskens–Pützstück [LLP25] gave another treatment of distributivity in the context of normed categories, which e.g. also captures examples in global equivariant homotopy theory. This is not possible with [NS22] since the global orbit category (for finite groups) **Glo** is not atomic orbital. On the other hand, the [LLP25] version is also more restrictive in the sense that only normed setups are allowed, so in the equivariant world only the $\mathbb{N}_\infty$-operads are permitted and not all G-∞-operads, which is the reason we are using [NS22]. It is expected that the [LLP25] distributivity agrees with [NS22] in the settings where both make sense but there is currently no known proof of this [LLP25, Remark 3.15].

Non-equivariantly, a monoidal ∞-category $(\mathscr{C}, \otimes)$ is called *distributive/monoidally cocomplete* [ACB19, Definition 2.4] if the tensor product $-\otimes-$ commutes with colimits in each variable. A shorter way of phrasing this is to demand for two functors $F\colon I \to \mathscr{C}$, $G\colon J \to \mathscr{C}$ and their associated colimit diagrams $I^{\triangleright}, J^{\triangleright} \to \mathscr{C}$ that the composition

$$(I\times J)^{\triangleright} \longrightarrow I^{\triangleright}\times J^{\triangleright} \longrightarrow \mathscr{C}\times\mathscr{C} \xrightarrow{\otimes} \mathscr{C}$$

is a colimit diagram. Indeed, the cone point of this composite is $\operatorname{colim}_I F \otimes \operatorname{colim}_J G$ while the composite restricted to $I \times J$ is $F\otimes G$, so this composite being a colimit diagram means

$$\operatorname*{colim}_I \operatorname*{colim}_J F\otimes G \simeq \operatorname*{colim}_I F \otimes \operatorname*{colim}_J G.$$

Nardin realized that suitably parametrizing this leads to a slick parametrized version of distributivity. To state his definition we first need a suitable source category for indexed tensor products.

To define this, we need a parametrized version of fibers, which we will define as the images of parametrized straightening functors.

Proposition 2.3.2 ([BDG+16, Proposition 8.3])**.** Let $\underline{\mathscr{C}} \in \mathbf{Cat}_{G,\infty}$, then there is an equivalence

$$\underline{\mathrm{St}}\colon \ \mathrm{coCart}(\underline{\mathscr{C}}) \simeq \mathrm{Map}_{\mathbf{Cat}_\infty}(\mathscr{C}, \mathbf{Cat}_\infty) \simeq \mathrm{Map}_{\mathbf{Cat}_{G,\infty}}(\underline{\mathscr{C}}, \underline{\mathbf{Cat}}_{G,\infty})$$

induced by the classical straightening-unstraightening equivalence and the forgetful-cofree adjunction. The inverse is denoted by $\underline{\mathrm{Un}}$.

Definition 2.3.3. Let $F\colon \underline{\mathscr{C}} \to \underline{\mathscr{D}}$ be a coCartesian fibration of G-∞-categories and $d \in \underline{\mathscr{D}}$ lying over $G/H \in \mathbf{Orb}_G^{\mathrm{op}}$. Then, we call $\underline{\mathscr{C}}_{\underline{d}} = \underline{\mathrm{St}}(F)(d) \in \mathbf{Cat}_{H,\infty}$ the parametrized fiber over d.

Proposition 2.3.4. Let $F\colon \underline{\mathscr{C}} \to \underline{\mathscr{D}}$ be a coCartesian fibration of G-∞-categories and $d \in \underline{\mathscr{D}}$ lying over $G/H \in \mathbf{Orb}_G^{\mathrm{op}}$. Then, there is a pullback square

$$\begin{tikzcd} \underline{\mathscr{C}}_{\underline{d}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathscr{C}} \arrow[d, "F"] \\ \mathbf{Orb}_H^{\mathrm{op}} \arrow[r] & \underline{\mathscr{D}} \end{tikzcd}$$

where the bottom arrow is classified by $d \in \underline{\mathscr{D}}_H$ by the Yoneda Lemma [CLL23, Lemma 2.2.7].

Proof. The diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathbf{Cat}_\infty}(\mathbf{Orb}_H^{\mathrm{op}}, \mathbf{Cat}_\infty) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Map}_{\mathbf{Cat}_{G,\infty}}(\underline{\mathbf{Orb}}_H^{\mathrm{op}}, \underline{\mathbf{Cat}}_{G,\infty}) \arrow[d, "\text{Yoneda}"] \\
\mathrm{coCart}(\mathbf{Orb}_H^{\mathrm{op}}) \arrow[r, equal] & \mathbf{Cat}_{H,\infty}
\end{tikzcd}$$

commutes by [CLL23, Remark 2.2.15]. Let $T\colon \underline{\mathscr{C}} \times_{\underline{\mathscr{D}}} \mathbf{Orb}_H^{\mathrm{op}} \to \mathbf{Orb}_H^{\mathrm{op}}$ be the pullback of F. Then, the above diagram evaluated at $\mathrm{St}(T)$ becomes

$$\begin{tikzcd}
\mathrm{St}(T) \arrow[r, mapsto] \arrow[d, mapsto] & \underline{\mathrm{St}}(T) \arrow[d, mapsto] \\
G \arrow[r, dash, "\simeq"'] & \underline{\mathrm{St}}(T)(G/H) = \underline{\mathscr{C}}_{\underline{d}}
\end{tikzcd}$$

where $\underline{\mathrm{St}}(G/H) \simeq \underline{\mathscr{C}}_{\underline{d}}$ uses that $\underline{\mathbf{Orb}}_H^{\mathrm{op}} \to \underline{\mathscr{D}}$ is classified by $d \in \mathscr{D}_H$. $\square$

Remark 2.3.5. In particular, we recover Nardin-Shah's notion of parametrized fibers [NS22, Notation 2.3.1] at least in the setting when $F\colon \underline{\mathscr{C}} \to \underline{\mathscr{D}}$ is a coCartesian fibration. They define

$$\underline{\mathscr{C}}_{\underline{d}} = * \times_{\underline{\mathscr{D}}} \mathrm{Ar}^{\mathrm{cocart}}(\underline{\mathscr{D}}) \times_{\underline{\mathscr{D}}} \underline{\mathscr{C}}$$

which we identify with the aforementioned pullback (Proposition 2.3.4) using the equivalence $\mathrm{Ar}^{\mathrm{cocart}}(\underline{\mathscr{D}}) \simeq \underline{\mathscr{D}} \times_{\mathbf{Orb}_G^{\mathrm{op}}} \mathrm{Ar}(\mathbf{Orb}_G^{\mathrm{op}})$, see [Sha22, Lemma 2.23].

We will in particular need this in the setting of $\mathscr{O}$-monoidal G-∞-categories where the Segal conditions allow us to give a more convenient description of those parametrized fibers.

Theorem 2.3.6 ([NS22, Theorem 2.3.3])**.** Let $\underline{\mathscr{O}}^{\otimes} \in \mathbf{Op}_{G,\infty}^{\mathrm{NS}}$ and $\underline{\mathscr{C}}^{\otimes} \in \mathbf{Mon}_{\underline{\mathscr{O}}}^{\mathrm{NS}}(\mathbf{Cat}_{G,\infty})$. Let $o \in \underline{\mathscr{O}}^{\otimes}$ be an object over $[f\colon U \to G/H] \in \underline{\mathbb{F}}_{G,*}$ with $H \leq G$. Let $U \simeq \coprod_i G/H_i$ with $H_i \leq G$ be an orbit decomposition and suppose that o corresponds to $(o_i)_i$ under the equivalence $\underline{\mathscr{C}}_U^{\otimes} \simeq \prod_i \underline{\mathscr{C}}_{H_i}^{\otimes}$. Then, there is an equivalence

$$\underline{\mathscr{O}}_{\underline{o}}^{\otimes} \simeq f_* \coprod_i \underline{\mathscr{O}}_{\underline{o_i}}^{\otimes}$$

of H-∞-categories, equivalently, we coinduce up each of the $\underline{\mathscr{C}}_{\underline{o_i}}^{\otimes}$ to a H-∞-category and take their product.

Definition 2.3.7. Let $\underline{\mathscr{O}}^{\otimes} \in \mathbf{Op}_{G,\infty}^{\mathrm{NS}}$ and $\underline{\mathscr{C}}^{\otimes} \in \mathbf{Mon}_{\underline{\mathscr{O}}}^{\mathrm{NS}}(\mathbf{Cat}_{G,\infty})$ and let $U \to G/H$ be some map in $\mathbb{F}_G$ corresponding to a map in $\underline{\mathbb{F}}_{G,*}$ over G/H. Let $o \to o'$ be a coCartesian lift of that map. Then, we associate to it an indexed tensor product functor

$$\underline{\bigotimes}_{o \to o'} = \underline{\mathrm{St}}(\underline{\mathscr{C}}^{\otimes})(o \to o')\colon \underline{\mathscr{C}}_{\underline{o}}^{\otimes} \to \underline{\mathscr{C}}_{\underline{o'}}^{\otimes}$$

of H-∞-categories.

Now, we have the language to recall the definition of distributivity.

Definition 2.3.8 ([NS22, Definition 3.2.3, 3.2.4])**.** Let $\underline{\mathscr{O}}^{\otimes} \in \mathbf{Op}_{G,\infty}^{\mathrm{NS}}$.

(i) Let $f\colon U \to V$ be a map in $\mathbb{F}_G$ as well as $\underline{\mathscr{C}} \in \mathbf{Cat}_{(\mathbf{Orb}_G)_{/U},\infty}$ and $\underline{\mathscr{D}} \in \mathbf{Cat}_{(\mathbf{Orb}_G)_{/V},\infty}$. A $(\mathbf{Orb}_G)_{/V}$-functor $F\colon f_*\underline{\mathscr{C}} \to \underline{\mathscr{D}}$ is called distributive if for every pullback square

$$\begin{tikzcd}
U' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & V' \arrow[d, "g"] \\
U \arrow[r, "f"'] & V
\end{tikzcd}$$

in $\mathbb{F}_G$ and every $(\mathbf{Orb}_G)_{/U'}$-colimit diagram $p\colon \underline{I}^{\triangleright} \to g'^*\underline{\mathscr{C}}$ the $(\mathbf{Orb}_G)_{/V'}$-functor

$$(f'_*\underline{I})^{\triangleright} \longrightarrow f'_*(\underline{I}^{\triangleright}) \xrightarrow{f'_*p} f'_*g'^*\underline{\mathscr{C}} \xrightarrow{\simeq,\mathrm{BC}} g^*f_*\underline{\mathscr{C}} \xrightarrow{g^*F} g^*\underline{\mathscr{D}}$$

is an $(\mathbf{Orb}_G)_{/V'}$-colimit-diagram.

(ii) Let $\underline{\mathscr{C}}^{\otimes} \in \mathbf{Mon}^{\mathrm{NS}}_{\underline{\mathscr{O}}}(\mathbf{Cat}_{G,\infty})$ and suppose that for all $H \leq G$ and $o' \in \underline{\mathscr{O}}^{\otimes}_H$ the parametrized fiber $\underline{\mathscr{C}}^{\otimes}_{o'} \in \mathbf{Cat}_{H,\infty}$ is H-cocomplete. Then, $\underline{\mathscr{C}}^{\otimes}$ is said to be ($\underline{\mathscr{O}}$-)distributive if for every coCartesian lift $o \to o'$ of a map in $\mathbb{F}_{G,*}$ over some $G/H \in \mathbf{Orb}_G$ corresponding to $f\colon U \to G/H$ the functor $\underline{\bigotimes}_{o\to o'}\colon \underline{\mathscr{C}}^{\otimes}_{\underline{o}} \to \underline{\mathscr{C}}^{\otimes}_{\underline{o}'}$ is distributive.

Part (ii) is well-defined: If $U \simeq \coprod_i G/H_i$ with $H_i \leq G$ is an orbit decomposition, then Theorem 2.3.6 provides an equivalence $\underline{\mathscr{C}}^{\otimes}_{\underline{o}} \simeq f_*\coprod_i \underline{\mathscr{C}}^{\otimes}_{\underline{o}_i}$, so the source of $\underline{\bigotimes}_{o\to o'}\colon \underline{\mathscr{C}}^{\otimes}_{\underline{o}} \to \underline{\mathscr{C}}^{\otimes}_{\underline{o}'}$ is really coinduced up as demanded in (i).

For an (ordinary) symmetric monoidal ∞-category the distributivity of

$$\bigotimes_{\langle 2\rangle\to\langle 1\rangle} = -\otimes -\colon \mathscr{C}^{\otimes} \times \mathscr{C}^{\otimes} \to \mathscr{C}^{\otimes}$$

unravels as exactly the classical distributivity notion mentioned in the introduction of this subsection.

Theorem 2.3.9 ([Nar17, Corollary 3.28])**.** There is a (unique) distributive G-symmetric monoidal structure on $\underline{\mathbf{Sp}}_G$.

Example 2.3.10. There exist G-cartesian G-symmetric monoidal structures $\underline{\mathcal{S}}^{\times}_G$ and $\underline{\mathbf{Cat}}^{\times}_{G,\infty}$ which are distributive [NS22, Proposition 3.2.5].

Lemma 2.3.11. Let $\underline{\mathscr{C}}^{\otimes} \to \underline{\mathbb{F}}_{G,*}$ be a distributive G-symmetric monoidal ∞-category, as well as $\underline{\mathscr{O}}^{\otimes} \in \mathbf{Op}^{\mathrm{NS}}_{G,\infty}$. Then, the pullback

$$\begin{tikzcd}
\underline{\mathscr{C}}^{\otimes} \times_{\underline{\mathbb{F}}_{G,*}} \underline{\mathscr{O}}^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathscr{C}}^{\otimes} \arrow[d] \\
\underline{\mathscr{O}}^{\otimes} \arrow[r] & \underline{\mathbb{F}}_{G,*}
\end{tikzcd}$$

is $\underline{\mathscr{O}}$-distributive.

Proof. First, this pullback can be taken in $\mathbf{Op}_{G,\infty}$ since the maps involved are maps of G-∞-operads. Moreover, we pull back a coCartesian fibration. Thus, $\underline{\mathscr{C}}^{\otimes} \times_{\underline{\mathbb{F}}_{G,*}} \underline{\mathscr{O}}^{\otimes}$ is indeed an $\underline{\mathscr{O}}$-monoidal G-∞-category. So now we can discuss distributivity.

The point is that distributivity is a notion that concerns the parametrized fibers but the parametrized fibers of $\underline{\mathscr{C}}^{\otimes} \times_{\underline{\mathbb{F}}_{G,*}} \underline{\mathscr{O}}^{\otimes} \to \underline{\mathscr{O}}^{\otimes}$ can be computed as certain parametrized fibers of $\underline{\mathscr{C}}^{\otimes} \to \underline{\mathbb{F}}_{G,*}$ and so the distributivity of $\underline{\mathscr{C}}^{\otimes}$ yields the $\underline{\mathscr{O}}$-distributivity of $\underline{\mathscr{C}}^{\otimes} \times_{\underline{\mathbb{F}}_G} \underline{\mathscr{O}}^{\otimes}$. □

The following projection formula result will be important to analyze the existence of indexed coproducts in parametrized left module categories (Proposition 3.3.11). This will in particular recover the classical projection formula for equivariant spectra using that $\underline{\mathbf{Sp}}^{\otimes}_G$ is distributive (Theorem 2.3.9).

Proposition 2.3.12. Let $\underline{\mathscr{C}}^{\otimes} \in \mathbf{Mon}^{\mathrm{NS}}_{\underline{\mathbb{F}}_{G,*}}(\mathbf{Cat}_{G,\infty})$ be distributive and $H \leq K \leq G$. For $A \in \underline{\mathscr{C}}^{\otimes}_K$ and $X \in \underline{\mathscr{C}}^{\otimes}_H$ we have $\mathrm{Ind}^K_H(\mathrm{Res}^K_H A \otimes X) \simeq A \otimes \mathrm{Ind}^K_H X$.

Proof. We consider $\mathbf{Orb}_H^{\mathrm{op}} \to \mathbf{Orb}_K^{\mathrm{op}}$ as an K-∞-category. By the Yoneda Lemma, the objects A, X correspond to K-functors

$$\underline{A}\colon \mathbf{Orb}_K^{\mathrm{op}} \to \underline{\mathscr{C}_K} \quad \text{and} \quad \underline{X}\colon \mathbf{Orb}_H^{\mathrm{op}} \to \underline{\mathscr{C}_K}.$$

Taking the corresponding colimit cones we obtain colimit diagrams $(\mathbf{Orb}_K^{\mathrm{op}})^{\trianglerighteq}, (\mathbf{Orb}_H^{\mathrm{op}})^{\trianglerighteq} \to \underline{\mathscr{C}_K}$. For distributivity (Definition 2.3.8) we consider the pullback square

$$\begin{tikzcd} G/K \amalg G/K \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & G/K \arrow[d, equal] \\ G/K \amalg G/K \arrow[r] & G/K \end{tikzcd}$$

in $\mathbb{F}_G$ and the parametrized colimit diagram

$$p\colon \left(\mathbf{Orb}_K^{\mathrm{op}}, \mathbf{Orb}_H^{\mathrm{op}}\right)^{\trianglerighteq} \to \underline{\mathscr{C}}^{\otimes}_{\underline{G/K \amalg G/K}} \simeq \left(\underline{\mathscr{C}_K}, \underline{\mathscr{C}_K}\right).$$

By distributivity the composite

$$\left(\mathbf{Orb}_K^{\mathrm{op}} \times \mathbf{Orb}_H^{\mathrm{op}}\right)^{\trianglerighteq} \longrightarrow \mathbf{Orb}_K^{\trianglerighteq,\mathrm{op}} \times \mathbf{Orb}_H^{\trianglerighteq,\mathrm{op}} \longrightarrow \underline{\mathscr{C}}_{\underline{K}}^{\otimes} \times \underline{\mathscr{C}}_{\underline{K}}^{\otimes} \xrightarrow{\ \otimes\ } \underline{\mathscr{C}}_{\underline{K}}^{\otimes}$$

is a K-colimit diagram which unravelled means

$$\underset{\mathbf{Orb}_K^{\mathrm{op}} \times \mathbf{Orb}_H^{\mathrm{op}}}{\underline{\mathrm{colim}}} (\underline{A} \otimes \underline{X}) \simeq \underset{\mathbf{Orb}_K^{\mathrm{op}}}{\underline{\mathrm{colim}}}\, \underline{A} \otimes \underset{\mathbf{Orb}_H^{\mathrm{op}}}{\underline{\mathrm{colim}}}\, \underline{X}.$$

where $\underline{\mathrm{colim}}$ denotes K-colimits. For the rest of the proof we want to verify that evaluating this on $K/K \in \mathbf{Orb}_K$ precisely yields the desired formula $\mathrm{Ind}_H^K(\mathrm{Res}_H^K A \otimes X) \simeq A \otimes \mathrm{Ind}_H^K X$.

Let us discuss the right side. By definition, the functors

$$\underset{\mathbf{Orb}_H^{\mathrm{op}}}{\underline{\mathrm{colim}}}\colon \underline{\mathrm{Fun}}_K\left(\mathbf{Orb}_H^{\mathrm{op}}, \underline{\mathscr{C}}_{\underline{K}}^{\otimes}\right) \to \underline{\mathscr{C}}_{\underline{K}}^{\otimes} \quad \text{and} \quad \underset{\mathbf{Orb}_K^{\mathrm{op}}}{\underline{\mathrm{colim}}}\colon \underline{\mathrm{Fun}}_K\left(\mathbf{Orb}_K^{\mathrm{op}}, \underline{\mathscr{C}}_{\underline{K}}^{\otimes}\right) \to \underline{\mathscr{C}}_{\underline{K}}^{\otimes}$$

are the K-left adjoints to the precomposition by the trivial map [Hil24b, Definition 3.1.1]. In particular, the second functor is equivalent to the identity functor which explains the equivalence $\underline{\mathrm{colim}}_{\mathbf{Orb}_K^{\mathrm{op}}} \underline{A} \simeq \underline{A}$. We now check that $\underline{\mathrm{colim}}_{\mathbf{Orb}_H^{\mathrm{op}}}$ evaluated on K/K is Ind_H^K. To make sense of this, we first of all compute the source of the functor as

$$\underline{\mathrm{Fun}}_K\left(\mathbf{Orb}_H^{\mathrm{op}}, \underline{\mathscr{C}}_{\underline{K}}^{\otimes}\right)_K \simeq \mathrm{Fun}_K\left(\mathbf{Orb}_H^{\mathrm{op}} \times \mathbf{Orb}_K^{\mathrm{op}}, \underline{\mathscr{C}_K}\right) \simeq \mathrm{Fun}_K\left(\mathbf{Orb}_H^{\mathrm{op}}, \underline{\mathscr{C}}_{\underline{K}}^{\otimes}\right)$$

by [CLL23, Corollary 2.2.9], so we are looking at the functor

$$\underset{\mathbf{Orb}_H^{\mathrm{op}}}{\underline{\mathrm{colim}}}\colon \mathrm{Fun}_K\left(\mathbf{Orb}_H^{\mathrm{op}}, \underline{\mathscr{C}}_{\underline{K}}^{\otimes}\right) \to \mathscr{C}_K^{\otimes}$$

whose right adjoint is informally described by sending $Y \in \mathscr{C}_K^{\otimes}$ to the natural transformation $(\mathrm{const}\,\mathrm{Res}_L^K Y)_{L \leq K}$. Thus, we compute

$$\begin{aligned} \mathrm{Map}_{\underline{\mathscr{C}}_K^{\otimes}}\left(\underset{\mathbf{Orb}_H^{\mathrm{op}}}{\underline{\mathrm{colim}}}\, \underline{X}, Y\right) &\simeq \mathrm{Map}_{\mathrm{Fun}_K\left(\mathbf{Orb}_H^{\mathrm{op}}, \underline{\mathscr{C}}_{\underline{K}}^{\otimes}\right)}\left(\underline{X}, (\mathrm{const}\,\mathrm{Res}_L^K Y)_{L \leq K}\right) \\ &= \mathrm{Map}_{\mathrm{Nat}\left(\mathrm{Map}_{\mathbf{Orb}_K}(-, K/H), \underline{\mathscr{C}}_{\underline{K}}^{\otimes}\right)}\left(\underline{X}, (\mathrm{const}\,\mathrm{Res}_L^K Y)_{L \leq K}\right) \\ &\simeq \mathrm{Map}_{\underline{\mathscr{C}}_H^{\otimes}}(X, \mathrm{Res}_H^K Y) \end{aligned}$$

by the Yoneda Lemma. By adjunction we conclude $\underline{\mathrm{colim}}_{\mathbf{Orb}_H^{\mathrm{op}}} \underline{X} \simeq \mathrm{Ind}_H^K X$. This demystifies the terms on the right side of the projection formula.

For the left side, an adjunction argument allows us to first compute a colimit over $\mathbf{Orb}_K^{\mathrm{op}}$ and then over $\mathbf{Orb}_H^{\mathrm{op}}$. The $\mathbf{Orb}_K^{\mathrm{op}}$-part only takes care of $\underline{A}$ which is unchanged as in the previous paragraph. However, we must apply Res_H^G to view $\underline{A} \otimes \underline{X}$ as an $\mathbf{Orb}_H^{\mathrm{op}}$-diagram and then the effect of $\underline{\mathrm{colim}}_{\mathbf{Orb}_H^{\mathrm{op}}}$ is Ind_H^K as demonstrated above. □

Let us end this subsection by recalling a parametrized version of operadic left Kan extensions.

Theorem 2.3.13 ([NS22, Proposition 4.3.3, Theorem 4.3.4])**.** Consider $\underline{\mathscr{O}}^{\otimes} \in \mathbf{Op}_{G,\infty}^{\mathrm{NS}}$ as well as $\underline{\mathscr{C}}^{\otimes}, \underline{\mathscr{D}}^{\otimes} \in \mathbf{Mon}_{\underline{\mathscr{O}}}^{\mathrm{NS}}(\underline{\mathbf{Cat}}_{G,\infty})$ with structure map $p \colon \underline{\mathscr{C}}^{\otimes} \to \underline{\mathscr{O}}^{\otimes}$. Let $\underline{\mathscr{D}}^{\otimes}$ be $\underline{\mathscr{O}}$-distributive.

(i) The restriction functor sits in an adjunction

$$\mathbf{Alg}_{\mathscr{C}/\mathscr{O}}(\mathscr{D}) \underset{p^*}{\overset{p_!}{\rightleftarrows}} \mathbf{Alg}_{\mathscr{O}/\mathscr{O}}(\mathscr{D})$$

The left adjoint $p_!$ is called operadic left Kan extension.

(ii) Let $\underline{\mathscr{O}}^{\otimes}$ have a single G-color and $F \in \mathbf{Alg}_{\mathscr{C}/\mathscr{O}}(\mathscr{D})$. Then, the underlying G-object of $p_!F$ is computed as the G-colimit of $F|_{\underline{\mathscr{C}}} \colon \underline{\mathscr{C}} \to \underline{\mathscr{D}}$.

Proof. Slightly more detailed, part (ii) combines the cited results with [Sha22, Proposition 10.8]. □

In particular, this shows that the parametrized colimit of a lax $\underline{\mathscr{O}}$-monoidal functor inherits an $\underline{\mathscr{O}}$-algebra structure.

3 Categorical Groundwork

3.1 Monoidality of Parametrized Slice Categories

Lurie uses simplicial methods to construct a monoidal structure on slice ∞-categories [Lur17, 2.2.2.1] which Antolín-Camarena–Barthel characterized by a universal property in [ACB19, Lemma 2.12].

We generalize this to the pattern setting through a model-independent approach. In particular, this yields a notion in parametrized higher category theory. For G-symmetric monoidal G-∞-categories and G-commutative algebras this has already been claimed in the literature without proof [HHK+24, Section A.5]. We are grateful to Jan Steinbrunner for suggesting the following to us.

Classically for an ∞-category $\mathscr{C}$, the over-slice of $\mathscr{C}$ over some object $c \in \mathscr{C}$ is the pullback

$$\begin{tikzcd} \mathscr{C}_{/c} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{C}^{[1]} \arrow[d, "t"] \\ * \arrow[r, "c"'] & \mathscr{C} \end{tikzcd}$$

This can be generalized to a higher algebraic setting by replacing the cotensor construction $\mathscr{C}^{[1]} = \mathrm{Fun}([1], \mathscr{C})$ in $\mathbf{Cat}_\infty$ by one in $\mathrm{coCart}(\mathbb{F}_*)$ and by replacing $c \in \mathscr{C}$ by an algebra. More generally, we can perform all of this for fibrous patterns, which is the purpose of this section.

This subsection uses the language of patterns more intensely than in all other sections and the reader can freely skip it by assuming the existence of a monoidal structure on G-∞-categories (Proposition 3.1.5) and the universal properties (Theorem 3.1.9, Proposition 3.1.11) it comes with.

The main ingredient to carry out the generalization to fibrous patterns is the cotensor which is well-studied in a more general setting, as we will now recall from [BHS24, Section 5.3]. Let $\mathscr{B} \in \mathbf{Cat}_\infty$ and $\mathscr{B}_0 \subset \mathscr{B}$ be a wide subcategory. Then, $\mathbf{Cat}_{\infty/\mathscr{B}}^{\mathscr{B}_0\text{-cocart}}$ denotes the subcategory of $\mathbf{Cat}_{\infty/\mathscr{B}}$ having coCartesian lifts over $\mathscr{B}_0$ and maps preserving those.

Proposition 3.1.1 ([BHS24, Construction 5.3.1, Proposition 5.3.2, Proposition 5.3.11(i)])**.** Consider $\mathscr{B} \in \mathbf{Cat}_\infty$ and a wide subcategory $\mathscr{B}_0 \subset \mathscr{B}$ as well as $\mathscr{C} \in \mathbf{Cat}_\infty$ and $(\mathscr{E} \to \mathscr{B}) \in \mathbf{Cat}_{\infty/\mathscr{B}}^{\mathscr{B}_0\text{-cocart}}$.

(i) Then, $\mathbf{Cat}_{\infty/\mathscr{B}}^{\mathscr{B}_0\text{-cocart}} \in \mathbf{LMod}_{\mathbf{Cat}_\infty}(\mathbf{Cat}_\infty^\times)$ with tensoring given by

$$\mathbf{Cat}_\infty \times \mathbf{Cat}_{\infty/\mathscr{B}}^{\mathscr{B}_0\text{-cocart}} \to \mathbf{Cat}_{\infty/\mathscr{B}}^{\mathscr{B}_0\text{-cocart}}, \ (\mathscr{C}, \mathscr{E} \to \mathscr{B}) \mapsto (\mathscr{E} \times \mathscr{C} \to \mathscr{B}).$$

(ii) The functor $- \otimes (\mathscr{E} \to \mathscr{B})\colon \mathbf{Cat}_\infty \to \mathbf{Cat}_{\infty/\mathscr{B}}^{\mathscr{B}_0\text{-cocart}}$ is left adjoint to $\mathrm{Fun}_{/\mathscr{B}}^{\mathscr{B}_0\text{-cocart}}(\mathscr{E} \to \mathscr{B}, -)$.

(iii) The functor $\mathscr{C} \otimes -\colon \mathbf{Cat}_{\infty/\mathscr{B}}^{\mathscr{B}_0\text{-cocart}} \to \mathbf{Cat}_{\infty/\mathscr{B}}^{\mathscr{B}_0\text{-cocart}}$ is left adjoint to a cotensoring, with the cotensor of $\mathscr{C} \in \mathbf{Cat}_\infty$ and $\mathscr{E} \to \mathscr{B}$ given by

$$\mathscr{E}_{/\mathscr{B}}^{\mathscr{C}} = \mathrm{Fun}(\mathscr{C}, \mathscr{E}) \times_{\mathrm{Fun}(\mathscr{C}, \mathscr{B})} \mathscr{B} \to \mathscr{B}$$

where the pullback is formed along the constant diagram $\mathscr{B} \to \mathrm{Fun}(\mathscr{C}, \mathscr{B})$.

(iv) Let $\mathscr{O} \in \mathbf{AlgPatt}$, $\mathscr{P} \in \mathbf{Fbrs}(\mathscr{O})$ and $\mathscr{C} \in \mathbf{Cat}_\infty$. Then, the cotensor $\mathscr{P}_{/\mathscr{O}}^{\mathscr{C}}$ in $\mathbf{Cat}_{\infty/\mathscr{O}}^{\text{int-cocart}}$ is again a fibrous pattern.

(v) Let $\mathscr{O}$ be a sound pattern, $\mathscr{D}^\otimes \in \mathbf{Seg}_{\mathscr{O}}(\mathbf{Cat}_\infty)$, $\mathscr{C} \in \mathbf{Cat}_\infty$. Then, $(\mathscr{D}^\otimes)_{/\mathscr{O}}^{\mathscr{C}} \in \mathbf{Seg}_{\mathscr{O}}(\mathbf{Cat}_\infty)$.

Proof. Part (v) follows from (iii) and (iv) since $(\mathscr{D}^\otimes)_{/\mathscr{O}}^{\mathscr{C}} \to \mathscr{O}$ is a fibrous pattern and a coCartesian fibration, so it is a Segal $\mathscr{O}$-∞-category given the soundness [BHS24, Observation 4.1.10]. □

With this cotensor we may first define a slice construction in great generality of which in particular the fibrous pattern version of it will be of interest to us in this paper.

Definition 3.1.2. Let $\mathscr{O} \in \mathbf{Cat}_\infty$ and $(\mathscr{D} \to \mathscr{O}) \in \mathbf{Cat}_{\infty/\mathscr{O}}$ and $A\colon \mathscr{O} \to \mathscr{D}$ in $\mathbf{Cat}_\infty$. Then, we define $\mathscr{D}_{/A} = \mathscr{D}_{/\mathscr{O}}^{[1]} \times_{\mathscr{D}} \mathscr{O} \to \mathscr{O}$, the (over)-slice ∞-category of $\mathscr{D}$ over A as the pullback

$$\begin{tikzcd}
\mathscr{D}_{/A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{D}_{/\mathscr{O}}^{[1]} \arrow[d, "\mathrm{target}\circ\mathrm{pr}_{\mathrm{Ar}(\mathscr{D})}"] \\
\mathscr{O} \arrow[r, "A"'] & \mathscr{D}
\end{tikzcd}$$

in $\mathbf{Cat}_\infty$ where the left leg is the structure map.

Lemma 3.1.3. Let $\mathscr{O} \in \mathbf{Cat}_\infty$ and $(\mathscr{D} \to \mathscr{O}) \in \mathbf{Cat}_{\infty/\mathscr{O}}$ and $A\colon \mathscr{O} \to \mathscr{D}$ in $\mathbf{Cat}_\infty$. For $o \in \mathscr{O}$ we have $(\mathscr{D}_{/A})_o \simeq (\mathscr{D}_o)_{/A(o)}$.

Proof. Recall that $(\mathscr{D}_{/A})_o$ denotes the fiber of $\mathscr{D}_{/A} \to \mathscr{O}$ at $o \in \mathscr{O}$. Consider the following diagram:

$$\begin{tikzcd}
(\mathscr{D}_{/A})_o \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{D}^{[1]}_{/\mathscr{O}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{O} \arrow[d] \\
\mathscr{D}_{/A(o)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}(\mathscr{D}) \arrow[r] \arrow[d] & \mathrm{Ar}(\mathscr{O}) \\
* \arrow[r, "A(o)"'] & \mathscr{D} &
\end{tikzcd}$$

Here, the top right square is a pullback square by construction of the cotensor (Proposition 3.1.1 (iii)), the bottom square is a pullback by definition of slice categories and the left composite rectangle is a pullback square by definition of $(\mathscr{D}_{/A})_o$. In particular, the top left square is a pullback square by pullback pasting.

Thus, the top composite pullback rectangle yields $(\mathscr{D}_{/A})_o \simeq \mathscr{D}_{/A(o)} \times_{\mathrm{Ar}(\mathscr{O})} \mathscr{O}$. We further factor $\mathscr{D}_{/A(o)} \to \mathrm{Ar}(\mathscr{O})$ through $\mathscr{O}_{/o}$ and with it factor the pullback into two steps as follows:

$$\begin{tikzcd}
(\mathscr{D}_{/A})_o \simeq \mathscr{D}_{/A(o)} \times_{\mathrm{Ar}(\mathscr{O})} \mathscr{O} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{O} \times_{\mathrm{Ar}(\mathscr{O})} \mathscr{O}_{/o} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathscr{O} \arrow[d] \arrow[dd, bend left, equal] \\
\mathscr{D}_{/A(o)} \arrow[r] & \mathscr{O}_{/o} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}(\mathscr{O}) \arrow[d] \\
& * \arrow[r, "o"'] & \mathscr{O}
\end{tikzcd}$$

The bottom square is also a pullback by construction and the right composite arrow is $\mathrm{id}_{\mathscr{O}}$. Thus, $\mathscr{O} \times_{\mathrm{Ar}(\mathscr{O})} \mathscr{O}_{/o} \simeq *$. Therefore,

$$\begin{aligned}
(\mathscr{D}_{/A})_o &\simeq \mathscr{D}_{/A(o)} \times_{\mathscr{O}_{/o}} * \\
&\simeq (\mathrm{Ar}(\mathscr{D}) \times_{\mathscr{D}} *) \times_{\mathrm{Ar}(\mathscr{O}) \times_{\mathscr{O}} *} (\mathrm{Ar}(*) \times_* *) \\
&\simeq (\mathrm{Ar}(\mathscr{D}) \times_{\mathrm{Ar}(\mathscr{O})} \mathrm{Ar}(*)) \times_{\mathscr{D} \times_{\mathscr{O}} *} (* \times_* *) \\
&\simeq \mathrm{Ar}(\mathscr{D}_o) \times_{\mathscr{D}_o} * \\
&\simeq (\mathscr{D}_o)_{/A(o)}
\end{aligned}$$

as desired. □

Example 3.1.4. Let $\underline{\mathscr{D}} \to \mathbf{Orb}_G^{\mathrm{op}}$ be a G-∞-category and $d\colon \mathbf{Orb}_G^{\mathrm{op}} \to \underline{\mathscr{D}}$ be an G-object therein. Then, $\underline{\mathscr{D}}_{/d}$ recovers the usual parametrized slice [Sha22, Notation 4.29] and for $H \leq G$ we obtain $(\underline{\mathscr{D}}_{/d})_H \simeq (\underline{\mathscr{D}}_H)_{/d_H}$ by the previous Lemma (Lemma 3.1.3).

The most important example for us will be slicing Segal $\mathscr{O}$-∞-categories over an algebra. We now show that these remain $\mathscr{O}$-Segal.

Proposition 3.1.5. Let $\mathscr{O}^{\otimes} \in \mathbf{AlgPatt}$ with $\mathscr{D}^{\otimes} \in \mathbf{Fbrs}(\mathscr{O}^{\otimes})$ and $A \in \mathbf{Alg}_{\mathscr{O}}(\mathscr{D})$.

(i) Then, $\mathscr{D}^{\otimes}_{/A} \to \mathscr{O}^{\otimes}$ is also a fibrous pattern.

(ii) If $\mathscr{D}^{\otimes} \to \mathscr{O}^{\otimes}$ is furthermore a coCartesian fibration, then so is $\mathscr{D}^{\otimes}_{/A} \to \mathscr{O}^{\otimes}$.

In particular, if $\mathscr{O}^{\otimes}$ is soundly extendable and $\mathscr{D}^{\otimes} \in \mathbf{Seg}_{\mathscr{O}}(\mathbf{Cat}_\infty)$, then $\mathscr{D}^{\otimes}_{/A} \in \mathbf{Seg}_{\mathscr{O}}(\mathbf{Cat}_\infty)$.

Proof. The 'in particular' part is a consequence of [BHS24, Remark 4.2.7].

In the course of this proof we will need to find coCartesian edges of certain functors. For the convenience of the reader, we will write informal proofs in terms of diagrams which are formalized through mapping spacifying the argument.

(i) Pullbacks in $\mathbf{Fbrs}(\mathcal{O}^{\otimes})$ which are computed in $\mathbf{Cat}_\infty$ [BHS24, Lemma 4.1.13], so we consider the pullback square

$$\begin{tikzcd} \mathcal{D}^{\otimes}_{/A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{D}^{\otimes})^{[1]}_{/\mathcal{O}^{\otimes}} \arrow[d] \\ \mathcal{O}^{\otimes} \arrow[r, "A"'] & \mathcal{D}^{\otimes} \end{tikzcd}$$

Since A is an $\mathcal{O}$-algebra, the bottom map is a map of fibrous patterns. Thus, it suffices to show that the right vertical map is one as well.

Let $o \to o'$ be an inert map in $\mathcal{O}^{\otimes}$ and $(d_1 \to d_2)$ be a map in $(\mathcal{D}^{\otimes})^{[1]}_{/\mathcal{O}^{\otimes}}$. Let $d_1 \to d_1'$ and $d_2 \to d_2'$ be coCartesian lifts of $o \to o'$, which exist because $\mathcal{D}^{\otimes} \in \mathbf{Fbrs}(\mathcal{O}^{\otimes})$. Then, there is a unique factorization

$$\begin{tikzcd} & d_1' \arrow[dr, dashed, "\exists!"] & \\ d_1 \arrow[ur] \arrow[r] & d_2 \arrow[r] & d_2' \end{tikzcd} \qquad \begin{tikzcd} & o' \arrow[dr, equal] & \\ o \arrow[ur] \arrow[r, equal] & o \arrow[r] & o' \end{tikzcd}$$

lying over the diagram on the right. In particular, the dashed arrow lies over $\mathrm{id}_{o'}$. Thus, we have constructed an edge $(d_1 \to d_2) \to (d_1' \to d_2')$, which we claim to be a coCartesian lift. Indeed, consider

$$\begin{tikzcd} & (d_1' \to d_2') \arrow[dr, dashed] & \\ (d_1 \to d_2) \arrow[ur] \arrow[rr] & & (d_1'' \to d_2'') \end{tikzcd} \qquad \begin{tikzcd} & o' \arrow[dr] & \\ o \arrow[ur] \arrow[rr] & & o'' \end{tikzcd}$$

with the left diagram lying over the right. We need to show that there exists a unique dashed arrow. The arrows $d_1' \to d_1''$ resp. $d_2' \to d_2''$ exist uniquely for the $(-)_1$ resp. the $(-)_2$ parts of the diagrams. So we need to show that

$$\begin{tikzcd} d_1' \arrow[r] \arrow[d] & d_2' \arrow[d] \\ d_1'' \arrow[r] & d_2'' \end{tikzcd}$$

commutes. For this consider the following left diagram lying over the right diagram.

$$\begin{tikzcd} & d_1' \arrow[dr, dashed, "\exists!"] & \\ d_1 \arrow[ur] \arrow[r] & d_1'' \arrow[r] & d_2'' \end{tikzcd} \qquad \begin{tikzcd} & o' \arrow[dr] & \\ o \arrow[ur] \arrow[r] & o'' \arrow[r, equal] & o'' \end{tikzcd}$$

By the universal property of the coCartesian edge $d_1 \to d_1'$ there exists a unique dashed arrow but each of the two composites in our square above fits. Thus, the square must commute.

This shows that the coCartesian lift of an inert edge of $(\mathcal{D}^{\otimes})^{[1]}_{/\mathcal{O}^{\otimes}} \to \mathcal{O}^{\otimes}$ agrees with pointwise coCartesian lifts of $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$. In particular, $(\mathcal{D}^{\otimes})^{[1]}_{/\mathcal{O}^{\otimes}} \to \mathcal{D}^{\otimes}$ must preserve inert edges, i.e. is a map in $\mathbf{Fbrs}(\mathcal{O}^{\otimes})$.

(ii) We compute the coCartesian lift over every edge.

Let $o \to o'$ be a map in $\mathscr{O}^{\otimes}$ and $(d \to A(o))$ lie over o. Let $d \to d'$ be a coCartesian lift of $o \to o'$ starting in d provided by the coCartesian fibration $\mathscr{D}^{\otimes} \to \mathscr{O}^{\otimes}$ and let $A(o) \to A'$ be the coCartesian lift of $o \to o'$ starting in $A(o)$. By the universal property of coCartesian fibrations there is a unique factorization

$$\begin{tikzcd} & A' \arrow[dr, dashed, "\exists!"] & \\ A(o) \arrow[ur] \arrow[rr, "A(o\to o')"'] & & A(o') \end{tikzcd} \qquad \begin{tikzcd} & o' \arrow[dr, "o'"] & \\ o \arrow[ur] \arrow[rr] & & o' \end{tikzcd}$$

where the left diagram lies over the right. By the universal property of coCartesian edges, there is a unique factorization

$$\begin{tikzcd} & d' \arrow[dr, dashed, "\exists!"] & \\ d \arrow[ur] \arrow[r] & A(o) \arrow[r] & A' \end{tikzcd} \qquad \begin{tikzcd} & o' \arrow[dr, equal] & \\ o \arrow[ur] \arrow[r, equal] & o \arrow[r] & o' \end{tikzcd}$$

where the left diagram lies over the right. Altogether, this provides a map

$$\begin{tikzcd} d \arrow[rr] \arrow[d] & & d' \arrow[dl] \arrow[d, dashed] \\ A(o) \arrow[r] & A' \arrow[r] & A(o'), \end{tikzcd}$$

which we claim to be the coCartesian lift of $o \to o'$ starting in $(d \to A(o))$. Note that the diagram commutes by construction. Moreover, the bottom map is $A(o \to o')$ by definition, so this is really a map in the slice category.

Indeed, consider

$$\begin{tikzcd} & o' \arrow[dr] & \\ o \arrow[ur] \arrow[rr] & & o'' \end{tikzcd}$$

and some $(d \to A(o)) \to (d'' \to A(o''))$. We need to find a unique factorization

$$\begin{tikzcd} & (d' \to A' \to A(o')) \arrow[dr, dashed] & \\ (d \to A(o)) \arrow[ur] \arrow[rr] & & (d'' \to A(o'')) \end{tikzcd}$$

lying over that previous triangle. By the universal property of the coCartesian lift $d \to d'$, there is a unique map $d' \to d''$ making the $d \to d' \to d''$ part commute. Moreover, $A(o') \to A(o'')$ must be $A(o' \to o'')$ for this to be a map in the slice category. By construction it also makes the $A(o) \to A(o') \to A(o'')$ part commute. It remains to check that

$$\begin{tikzcd} d' \arrow[r] \arrow[d] & A(o') \arrow[d] \\ d'' \arrow[r] & A(o'') \end{tikzcd}$$

commutes. For this we consider the diagram

$$\begin{tikzcd} & d' \arrow[dr, dashed, "\exists!"] & \\ d \arrow[ur] \arrow[r] & d'' \arrow[r] & A(o'') \end{tikzcd} \qquad \begin{tikzcd} & o' \arrow[dr] & \\ o \arrow[ur] \arrow[r] & o'' \arrow[r, equal] & o'' \end{tikzcd}$$

where the left diagram lies over the right.

By all the commutative diagrams already provided in that triangle of arrows above, we see that both ways of going the square are allowed for the dashed arrow. By uniqueness it follows that they agree. □

Corollary 3.1.6. Let $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $\mathscr{D}^{\otimes} \in \mathbf{Mon}_{\mathscr{O}}(\mathbf{Cat}_{\infty})$ with $A \in \mathbf{Alg}_{\mathscr{O}}(\mathscr{D})$. Let $o \to o'$ be a map in $\mathscr{O}^{\otimes}$ and $(X_o \to A(o)) \in (\mathscr{D}^{\otimes}_o)_{/A(o)}$, then the $(o \to o')$-indexed tensor product in $\mathscr{D}^{\otimes}_{/A}$ is given by

$$\bigotimes_{o\to o'}(X_o \to A(o)) \simeq \left(\bigotimes_{o\to o'} X_o \to \bigotimes_{o\to o'} A(o) \xrightarrow{\mu} A(o')\right)$$

where μ is the algebra multiplication.

Proof. Recall that $\bigotimes_{o\to o'}(X_o \to A(o))$ denotes the coCartesian edge starting in $(X_o \to A(o))$ lying over $o \to o'$. In the proof of Proposition 3.1.5 we described the coCartesian edges of $\mathscr{D}^{\otimes}_{/A}$, so the indexed tensor product can be immediately read off. □

Example 3.1.7. More concretely, if $\mathscr{C}^{\otimes} \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_{\infty})$ and $A \in \mathbf{Alg}_{\mathrm{Span}(\mathbb{F}_G)}(\mathscr{C})$, then the indexed product is described in the following more familiar fashion:

(i) Let $\nabla\colon G/H \amalg G/H \to G/H$ be the fold map and $X \to A_H, Y \to A_H$ be maps in $\mathscr{C}_H$. Then,
$$\bigotimes_{\nabla}(X \to A_H, Y \to A_H) \simeq (X \otimes Y \to A_H \otimes A_H \to A_H).$$

(ii) Let $H \leq K$ and $G/H \to G/K$ be the projection map and $X \to A_H$ be a map in $\mathscr{C}_H$. Then,
$$\bigotimes_{G/H\to G/K}(X \to A_H) \simeq \left(N_H^K X \to N_H^K A_H \to A_K\right)$$
where N_H^K is given by the indexed tensor product $\bigotimes_{G/H\to G/K}$.

In particular, this is the usual tensor product on slice categories for $G = e$ which is exemplified in (i).

Proposition 3.1.8. Let $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ with $\mathscr{D}^{\otimes} \in \mathbf{Mon}_{\mathscr{O}}(\mathbf{Cat}_{\infty})$ and $A \in \mathbf{Alg}_{\mathscr{O}}(\mathscr{D})$. Then, the underlying G-∞-category of $\mathscr{D}^{\otimes}_{/A}$ is the slice G-∞-category $\underline{\mathscr{D}}_{/\underline{A}}$.

Proof. For the sake of the following computation, let us write $S = \mathrm{Span}(\mathbb{F}_G)$. We omit writing the map to $\mathbf{Orb}_G^{\mathrm{op}}$ and perform in $\mathrm{coCart}(\mathbf{Orb}_G^{\mathrm{op}})$ the following computation:

$$\begin{aligned}
\mathscr{D}^{\otimes}_{/A} \times_S \mathbf{Orb}_G^{\mathrm{op}} &\simeq \left((\mathscr{D}^{\otimes})^{[1]}_{/S} \times_{\mathscr{D}^{\otimes}} S)\right) \times_S \mathbf{Orb}_G^{\mathrm{op}} \\
&\simeq \left((\mathscr{D}^{\otimes})^{[1]}_{/S} \times_S \mathbf{Orb}_G^{\mathrm{op}}\right) \times_{\mathscr{D}^{\otimes}\times_S \mathbf{Orb}_G^{\mathrm{op}}} (S \times_S \mathbf{Orb}_G^{\mathrm{op}}) \\
&\simeq \left((\mathscr{D}^{\otimes})^{[1]}_{/S} \times_S \mathbf{Orb}_G^{\mathrm{op}}\right) \times_{\underline{\mathscr{D}}} \mathbf{Orb}_G^{\mathrm{op}} \\
&\simeq \left(\left(\mathrm{Ar}(\mathscr{D}^{\otimes}) \times_{\mathrm{Ar}(S)} S\right) \times_S \mathbf{Orb}_G^{\mathrm{op}}\right) \times_{\underline{\mathscr{D}}} \mathbf{Orb}_G^{\mathrm{op}} \\
&\simeq \left(\mathrm{Ar}(\mathscr{D}^{\otimes}) \times_{\mathrm{Ar}(S)} \mathbf{Orb}_G^{\mathrm{op}}\right) \times_{\underline{\mathscr{D}}} \mathbf{Orb}_G^{\mathrm{op}} \\
&\simeq \left(\mathrm{Ar}(\mathscr{D}^{\otimes}) \times_{\mathrm{Ar}(S)} \mathrm{Ar}(\mathbf{Orb}_G^{\mathrm{op}}) \times_{\mathrm{Ar}(\mathbf{Orb}_G^{\mathrm{op}})} \mathbf{Orb}_G^{\mathrm{op}}\right) \times_{\underline{\mathscr{D}}} \mathbf{Orb}_G^{\mathrm{op}} \\
&\simeq \left(\mathrm{Ar}(\mathscr{D}^{\otimes} \times_S \mathbf{Orb}_G^{\mathrm{op}}) \times_{\mathrm{Ar}(\mathbf{Orb}_G^{\mathrm{op}})} \mathbf{Orb}_G^{\mathrm{op}}\right) \times_{\underline{\mathscr{D}}} \mathbf{Orb}_G^{\mathrm{op}} \\
&\simeq \left(\mathrm{Ar}(\underline{\mathscr{D}}) \times_{\mathrm{Ar}(\mathbf{Orb}_G^{\mathrm{op}})} \mathbf{Orb}_G^{\mathrm{op}}\right) \times_{\underline{\mathscr{D}}} \mathbf{Orb}_G^{\mathrm{op}} \\
&\simeq \underline{\mathscr{D}}^{[1]}_{/\mathbf{Orb}_G^{\mathrm{op}}} \times_{\underline{\mathscr{D}}} \mathbf{Orb}_G^{\mathrm{op}},
\end{aligned}$$

which is the classical parametrized slice (Example 3.1.4). $\square$

So we are really putting an $\mathscr{O}$-monoidal structure on $\underline{\mathscr{D}}_{/A}$.

Theorem 3.1.9 (Universal Property of Monoidal Slice)**.** Let $\mathscr{O}^{\otimes} \in \mathbf{AlgPatt}, \mathscr{C}^{\otimes}, \mathscr{D}^{\otimes} \in \mathbf{Fbrs}(\mathscr{O}^{\otimes})$ as well as $A \in \mathbf{Alg}_{\mathscr{O}}(\mathscr{D})$ and $F \in \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})$. Let $p\colon \mathscr{C}^{\otimes} \to \mathscr{O}^{\otimes}$ denote the structure map. Then, there are equivalences:

(i) $\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes}_{/A}) \simeq \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})_{/A\circ p}$.

(ii) $\mathrm{Map}_{\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes},\mathscr{D}^{\otimes})}(F, A\circ p) \simeq \mathrm{Map}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})_{/\mathscr{D}^{\otimes}}}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes}_{/A})$.

Proof.

(i) We compute

$$\begin{aligned}
\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes}_{/A}) &\simeq \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}\left(\mathscr{C}^{\otimes}, (\mathscr{D}^{\otimes})^{[1]}_{/\mathscr{O}^{\otimes}} \times_{\mathscr{D}^{\otimes}} \mathscr{O}^{\otimes}\right) \\
&\simeq \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}\left(\mathscr{C}^{\otimes}, (\mathscr{D}^{\otimes}_{/\mathscr{O}})^{[1]}\right) \times_{\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes},\mathscr{D}^{\otimes})} \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes}, \mathscr{O}^{\otimes}) \\
&\simeq \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes} \times [1], \mathscr{D}^{\otimes}) \times_{\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes},\mathscr{D}^{\otimes})} * \\
&\simeq \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})^{[1]} \times_{\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes},\mathscr{D}^{\otimes})} * \\
&\simeq \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})_{/A\circ p}.
\end{aligned}$$

where we have used the tensor-cotensor adjunction (Proposition 3.1.1) a number of times. In the second equivalence we used that $\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})}(\mathscr{C}^{\otimes}, -)$ is a right adjoint (Proposition 3.1.1 (ii)) and thus preserves limits. In the third equivalence, we used a Yoneda-type argument to argue that the tensor-cotensor adjunction enriched to functor categories and the fourth equivalence is also by a Yoneda argument.

(ii) Using (i) we obtain

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O})}(\mathscr{C}^{\otimes},\mathscr{D}^{\otimes})}(F, A\circ p) &\simeq \{F\} \times_{\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O})}(\mathscr{C}^{\otimes},\mathscr{D}^{\otimes})} \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O})}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes})_{/A\circ p} \\
&\simeq \{F\} \times_{\mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O})}(\mathscr{C}^{\otimes},\mathscr{D}^{\otimes})} \mathrm{Fun}_{\mathbf{Fbrs}(\mathscr{O})}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes}_{/A}) \\
&\simeq \mathrm{Map}_{\mathbf{Fbrs}(\mathscr{O}^{\otimes})_{/\mathscr{D}^{\otimes}}}(\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes}_{/A})
\end{aligned}$$

as desired.

$\square$

Remark 3.1.10. Let $\mathcal{O}^{\otimes} \in \mathbf{Op}_{\infty}, \mathcal{D}^{\otimes} \in \mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_{\infty})$ and $A \in \mathbf{Alg}_{\mathcal{O}}(\mathcal{D})$. Then, our monoidal slice construction $\mathcal{D}^{\otimes}_{/A}$ agrees with Lurie's slice construction $\mathcal{D}_{/A_{\mathcal{O}}}$ in [Lur17, Notation 2.2.2.3] because our universal property (Theorem 3.1.9 (ii)) specializes to a universal property that Lurie's monoidal slice also satisfies for ordinary ∞-operads [ACB19, Lemma 2.12].

We end this section by giving a relation between operadic left Kan extensions and the slice monoidal structure. This is a generalization of [ACB19, Theorem 2.13] and is the abstract version of the universal property of multiplicative equivariant Thom spectra that is to come in later sections (Theorem 4.1.8).

Proposition 3.1.11. Let $\underline{\mathcal{O}}^{\otimes} \in \mathbf{Op}^{\mathrm{NS}}_{G,\infty}$ with $\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes} \in \mathbf{Mon}^{\mathrm{NS}}_{\underline{\mathcal{O}}}(\underline{\mathbf{Cat}}_{G,\infty})$ and $A \in \mathbf{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{D}})$. Let $p\colon \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{O}}^{\otimes}$ denote the structure map and $F \in \mathbf{Alg}_{\underline{\mathcal{C}}/\underline{\mathcal{O}}}(\underline{\mathcal{D}})$. Suppose that $\underline{\mathcal{D}}^{\otimes}$ is $\underline{\mathcal{O}}$-distributive. Then, there is an equivalence

$$\mathrm{Map}_{\mathbf{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{D}})}(p_!F, A) \simeq \mathrm{Map}_{\mathbf{Fbrs}(\underline{\mathcal{O}}^{\otimes})_{/\underline{\mathcal{D}}^{\otimes}}}(\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes}_{/A})$$

where we endow $\underline{\mathcal{D}}^{\otimes}_{/A}$ with the slice monoidal structure.

Proof. We compute

$$\begin{aligned}\mathrm{Map}_{\mathbf{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{D}})}(p_!F, A) &\simeq \mathrm{Map}_{\mathbf{Alg}_{\underline{\mathcal{C}}/\underline{\mathcal{O}}}(\underline{\mathcal{D}})}(F, A \circ p)\\ &\simeq \mathrm{Map}_{\mathbf{Fbrs}(\underline{\mathcal{O}}^{\otimes})_{/\underline{\mathcal{D}}^{\otimes}}}(F, \underline{\mathcal{D}}^{\otimes}_{/A} \to \underline{\mathcal{D}}^{\otimes})\end{aligned}$$

by operadic left Kan extension (Theorem 2.3.13) and the universal property of the monoidal slice category (Theorem 3.1.9). $\square$

Colloquially, a map $p_!F \to A$ corresponds to a lift

$$\begin{tikzcd} & \underline{\mathcal{D}}^{\otimes}_{/A} \arrow[d] \\ \underline{\mathcal{C}}^{\otimes} \arrow[ur, dashed] \arrow[r, "F"'] & \underline{\mathcal{D}}^{\otimes} \end{tikzcd}$$

over $\underline{\mathcal{O}}^{\otimes}$. This characterizes the operadic left Kan extension $p_!F$.

3.2 Microcosmic Parametrized Monoidal Straightening-Unstraightening

Let $X \in \mathcal{S}_G$, then (parametrized) straightening-unstraightening (Proposition 2.3.2) gives rise to an equivalence $(\underline{\mathcal{S}}_G)_{/X} \simeq \underline{\mathbf{PSh}}_G(X)$. For $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $X^{\otimes} \in \mathbf{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{S}}_G)$, both sides naturally enhance to $\mathcal{O}$-monoidal ∞-categories, namely via the slice monoidal structure (Proposition 3.1.5) and (parametrized) Day convolution (Theorem 3.2.2).

One naturally expects the equivalence to enhance to an $\mathcal{O}$-monoidal equivalence. Indeed, this statement has already been stated in the literature without proof [HHK+24, Theorem A.6.1]. We will not need a result as strong as this, and will put together a weaker statement on a *microcosmic* level with terminology inspired by [Ram22].

Here, the word 'microcosmic' essentially points to the result on the level of algebras. We don't claim much originality in this subsection, the claim will follow by putting together a number of results from the literature. Indeed, the following result from Stewart should already be viewed as a microcosmic parametrized monoidal straightening-unstraightening result. We also learned towards the end of the preparation of this article that our presentation is very similar to Cnossen's [Cno23, Proposition A.3], although we phrase our discussion in the more general setting of G-∞-operads.

Proposition 3.2.1 ([Ste25b, Corollary 1.52]). Let $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$. There exist cartesian G-symmetric monoidal structures on G-∞-categories with G-products for which there are equivalences

$$\mathbf{Alg}_{\underline{\mathcal{O}}}(\underline{\mathbf{Cat}}_{G,\infty}) \simeq \mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_{\infty}) \quad \text{and} \quad \mathbf{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{S}}_G) \simeq \mathbf{Mon}_{\mathcal{O}}(\mathcal{S})$$

where $\underline{\mathbf{Cat}}_{G,\infty}$ and $\underline{\mathcal{S}}_G$ are endowed with the cartesian G-symmetric monoidal structure.

Theorem 3.2.2 ([NS22, Section 3]). Consider $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ as well as $\mathscr{C}^{\otimes}, \mathscr{D}^{\otimes} \in \mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_{\infty})$. Suppose that $\mathscr{D}^{\otimes}$ is $\underline{\mathcal{O}}$-distributive. Then, there exists an $\mathcal{O}$-monoidal ∞-category $\mathrm{Fun}^{\mathcal{O}}(\mathscr{C}, \mathscr{D})^{\otimes}$ whose underlying G-∞-category is $\underline{\mathrm{Fun}}_G(\mathscr{C}, \mathscr{D})$ such that

$$\mathbf{Alg}_{\underline{\mathcal{O}}/\underline{\mathcal{O}}}\left(\underline{\mathrm{Fun}}^{\mathcal{O}}(\mathscr{C}, \mathscr{D})\right) \simeq \mathbf{Alg}_{\underline{\mathscr{C}}/\underline{\mathcal{O}}}(\underline{\mathscr{D}}).$$

For example, $\underline{\mathcal{S}}_G^{\times}$ is distributive (Example 2.3.10), so also its pullback along $\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_G)$ by Lemma 2.3.11, so for $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $X \in \mathbf{Mon}_{\mathcal{O}}(\mathcal{S})$ we find $\mathbf{PSh}_G^{\mathcal{O}}(X)^{\otimes} \in \mathbf{Mon}_{\mathcal{O}}(\mathcal{S})$ whose underlying G-∞-category is $\underline{\mathbf{PSh}}_G(X)$.

Corollary 3.2.3. Let $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$

(i) Let $\mathscr{C}^{\otimes} \in \mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_{\infty})$. Then, $\mathbf{Alg}_{\underline{\mathcal{O}}/\underline{\mathcal{O}}}\left(\underline{\mathrm{Fun}}^{\mathcal{O}}(\underline{\mathscr{C}}, \underline{\mathbf{Cat}}_{G,\infty})\right) \simeq \mathbf{Mon}_{\mathscr{C}}(\mathbf{Cat}_{\infty})$.

(ii) Let $X^{\otimes} \in \mathbf{Mon}_{\mathcal{O}}(\mathcal{S})$. Then, $\mathbf{Alg}_{\underline{\mathcal{O}}/\underline{\mathcal{O}}}(\underline{\mathbf{PSh}}_G^{\mathcal{O}}(X)) \simeq \mathbf{Mon}_X(\mathcal{S}) \simeq \mathbf{Mon}_{\mathcal{O}}(\mathcal{S})_{/X^{\otimes}}$.

Proof.

(i) By the universal property of Day convolution (Theorem 3.2.2) we compute

$$\begin{aligned}\mathbf{Alg}_{\underline{\mathcal{O}}/\underline{\mathcal{O}}}\left(\underline{\mathrm{Fun}}^{\mathcal{O}}(\underline{\mathscr{C}}, \underline{\mathbf{Cat}}_{G,\infty})\right) &\simeq \mathbf{Alg}_{\underline{\mathscr{C}}/\underline{\mathcal{O}}}(\underline{\mathcal{O}}^{\otimes} \times_{\underline{\mathbb{F}}_{G,*}} \underline{\mathbf{Cat}}^{\times}_{G,\infty}) \\ &\simeq \mathbf{Alg}_{\underline{\mathscr{C}}}(\underline{\mathbf{Cat}}_{G,\infty}) \\ &\simeq \mathbf{Mon}_{\mathscr{C}}(\mathbf{Cat}_{\infty})\end{aligned}$$

where we use Proposition 3.2.1 in the last step. Note in the first equivalence that we take the $\mathcal{O}$-monoidal version of $\underline{\mathbf{Cat}}_{G,\infty}$. Then, the second equivalence is for example via $\mathbf{Fbrs}(\underline{\mathcal{O}}^{\otimes}) \simeq \mathbf{Fbrs}(\underline{\mathbb{F}}_{G,*})_{/\mathcal{O}^{\otimes}}$ [BHS24, Corollary 4.1.17].

(ii) By the universal property of Day convolution (Theorem 3.2.2) and abusing $X \simeq X^{\mathrm{op}}$ we can thus compute

$$\begin{aligned}\mathbf{Alg}_{\underline{\mathcal{O}}/\underline{\mathcal{O}}}\left(\underline{\mathbf{PSh}}_G^{\mathcal{O}}(X)\right) &\simeq \mathbf{Alg}_{\underline{X}/\underline{\mathcal{O}}}(\underline{\mathcal{O}}^{\otimes} \times_{\underline{\mathbb{F}}_{G,*}} \underline{\mathcal{S}}_G^{\times}) \\ &\simeq \mathbf{Alg}_{\underline{X}}(\underline{\mathcal{S}}_G) \\ &\simeq \mathbf{Mon}_X(\mathcal{S})\end{aligned}$$

where we use Proposition 3.2.1 in the last step. Moreover, we get $\mathbf{Mon}_X(\mathcal{S}) \simeq \mathbf{Mon}_{\mathcal{O}}(\mathcal{S})_{/X^{\otimes}}$ because over spaces every edge is coCartesian. ☐

Corollary 3.2.4. Let $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $X^{\otimes} \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathcal{S})$. Then,

$$\mathbf{Alg}_{\underline{\mathcal{O}}}(\underline{\mathbf{PSh}}_G^{\mathcal{O}}(X)) \simeq \mathbf{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{S}}^G_{/X})$$

where $\underline{\mathcal{S}}^G_{/X}$ is endowed with the slice monoidal structure (Proposition 3.1.5).

Proof. Let us first remark that $\underline{\mathbf{PSh}}_G(X)^{\otimes} \times_{\mathbb{F}_{G,*}} \underline{\mathscr{O}}^{\otimes} \simeq \underline{\mathbf{PSh}}^{\mathscr{O}}(X)^{\otimes}$ where the second term implicitly uses $X^{\otimes} \times_{\mathbb{F}_{G,*}} \underline{\mathscr{O}}^{\otimes}$, i.e. the underlying $\underline{\mathscr{O}}$-monoidal space of $X^{\otimes}$. This follows by the explicit construction of Day convolution through norms [NS22, Definition 3.1.6] and one can compare them through the universal property of these norms [NS22, Proposition 3.1.7].

With this in our arsenal, we can write out

$$\begin{aligned}\mathbf{Alg}_{\underline{\mathscr{O}}}\left(\underline{\mathbf{PSh}}_G(X)\right) &\simeq \mathbf{Alg}_{\underline{\mathscr{O}}/\underline{\mathscr{O}}}\left(\underline{\mathbf{PSh}}_G(X)^{\otimes} \times_{\mathbb{F}_{G,*}} \underline{\mathscr{O}}^{\otimes}\right)\\ &\simeq \mathbf{Alg}_{\underline{\mathscr{O}}/\underline{\mathscr{O}}}(\underline{\mathbf{PSh}}_G^{\mathscr{O}}(X))\\ &\simeq \mathbf{Alg}_{\underline{\mathscr{O}}}(\underline{\mathcal{S}}^G)_{/X^{\otimes}}\\ &\simeq \mathbf{Alg}_{\underline{\mathscr{O}}}(\underline{\mathcal{S}}^G_{/X})\end{aligned}$$

where we use the previous result (Corollary 3.2.3) and the universal property of the slice monoidal structure (Theorem 3.1.9 (i)). $\square$

This is really the result that should be called microcosmic straightening-unstraightening. It is the equivalence you obtain by applying $\mathbf{Alg}_{\mathscr{O}}$ to a stronger macrocosmic straightening-unstraightening $\underline{\mathbf{PSh}}_G^{\mathscr{O}}(X)^{\otimes} \simeq (\underline{\mathcal{S}}^G_{/X})^{\otimes}$.

3.3 Monoidality of Parametrized Left Module Categories

We equivariantize Lurie's ∞-category of left modules $\mathbf{LMod}_A(\mathscr{C})$ via parametrized higher algebra to $\mathbf{LMod}_A^G(\mathscr{C})$. By adapting a criterion about coCartesian fibrations of Haugseng–Melani–Safronov [HMS22, Lemma A.1.8] to the parametrized setting we endow $\mathbf{LMod}_A^G(\mathscr{C})$ with a multiplicative structure. We then spend much effort in showing that this multiplicative structure is distributive in the sense of Nardin–Shah (Definition 2.3.8) culminating in the most technical proof of this article. Throughout, we are careful to make this work for more G-∞-operads than only the terminal one.

Let us first recall parts of the classical setting. Let $\mathscr{C}^{\otimes}$ be a symmetric monoidal ∞-category. There is a preferred map between ∞-operads $\mathbb{E}_1 \to \mathcal{LM}$ such that for $A \in \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})$ the pullback

$$\begin{tikzcd} \mathbf{LMod}_A(\mathscr{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Alg}_{\mathcal{LM}}(\mathscr{C}) \arrow[d] \\ * \arrow[r, "A"'] & \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C}) \end{tikzcd}$$

is the ∞-category of left A-modules in $\mathscr{C}$ [Lur17, Definition 4.2.1.13]. Recall that $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})$ is the ∞-category with objects (R, M) encoding an $\mathbb{E}_1$-algebra R that acts on some object $M \in \mathscr{C}$. Observe that $\mathbf{LMod}_A(\mathscr{C})$ is the fiber of a coCartesian fibration:

Lemma 3.3.1 (Lurie)**.** Let $\mathscr{C}^{\otimes}$ be a symmetric monoidal ∞-category whose underlying ∞-category $\mathscr{C}$ admits geometric realizations and whose tensor product distributes over these.

(i) Then, $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C}) \to \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})$ is a coCartesian fibration.

(ii) A coCartesian edge over $A \to B$ starting in (A, M) is given by $(A, M) \to (B, B \otimes_A M)$.

Proof. This is [Lur17, Lemma 4.5.3.6, Proposition 4.6.2.17]. $\square$

Lurie then proves the existence of a monoidal structure in suitable setups [Lur17, Corollary 4.8.5.20] and goes on with life.

Remark 3.3.2. This is not exactly what Lurie does in [Lur17, Definition 4.2.1.13]. His approach allows for more general inputs than a (symmetric) monoidal ∞-category $\mathscr{C}^{\otimes}$ but in case the input is a (symmetric) monoidal ∞-category $\mathscr{C}$, then his construction is equivalent to the one claimed above.

In that case, Lurie takes the same pullback square but with $\mathrm{Fun}^{\mathrm{lax}}_{/\mathbb{E}_1}(\mathcal{LM}^{\otimes}, \mathscr{C}^{\otimes} \times_{\mathbb{F}_*} \mathbb{E}_1^{\otimes})$ instead of $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})$ [Lur17, Example 4.2.1.16]. We show that these are actually equivalent. Consider the composite of pullback squares

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{lax}}_{/\mathbb{E}_1}(\mathcal{LM}^{\otimes}, \mathscr{C}^{\otimes} \times_{\mathbb{F}_*} \mathbb{E}_1^{\otimes}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{\mathrm{lax}}_{/\mathbb{F}_*}(\mathcal{LM}^{\otimes}, \mathscr{C}^{\otimes} \times_{\mathbb{F}_*} \mathbb{E}_1^{\otimes}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{\mathrm{lax}}_{/\mathbb{F}_*}(\mathcal{LM}^{\otimes}, \mathscr{C}^{\otimes}) \arrow[d] \\
* \arrow[r] & \mathrm{Fun}^{\mathrm{lax}}_{/\mathbb{F}_*}(\mathcal{LM}^{\otimes}, \mathbb{E}_1^{\otimes}) \arrow[r] & \mathrm{Fun}^{\mathrm{lax}}_{/\mathbb{F}_*}(\mathcal{LM}^{\otimes}, \mathbb{F}_*)
\end{tikzcd}$$

where the left pullback is for example [Lur17, Definition 2.1.3.1]. The right square is a pullback square since $\mathrm{Fun}_{\mathbf{Op}_\infty}(\mathcal{LM}^{\otimes}, -)$ preserves pullbacks. We conclude that the composite rectangle is a pullback diagram and since $\mathrm{Fun}^{\mathrm{lax}}_{/\mathbb{F}_*}(\mathcal{LM}^{\otimes}, \mathbb{F}_*) \simeq *$, we obtain

$$\mathrm{Fun}^{\mathrm{lax}}_{/\mathbb{E}_1}(\mathcal{LM}^{\otimes}, \mathscr{C}^{\otimes} \times_{\mathbb{F}_*} \mathbb{E}_1^{\otimes}) \simeq \mathrm{Fun}^{\mathrm{lax}}_{/\mathbb{F}_*}(\mathcal{LM}^{\otimes}, \mathscr{C}^{\otimes}) = \mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})$$

demonstrating that our exposition agrees with Lurie's.

Our idea is to refine the pullback square in two ways, namely in a parametrized direction and in a higher algebra direction.

Construction 3.3.3. Let $\mathscr{C}^{\otimes} \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_\infty)$ with $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $A \in \mathbf{Alg}_{\mathscr{O}\otimes\mathbb{E}_1}(\mathscr{C})$ corresponding to a map $\mathscr{O}^{\otimes} \to \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}$ in $\mathbf{Op}_{G,\infty}$ by adjunction (Theorem 2.2.4). Furthermore, note that $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \to \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}$ is a map in $\mathbf{Op}_{G,\infty}$ since they have the pointwise monoidal structure (Theorem 2.2.4). We denote by $\mathbf{LMod}^G_A(\mathscr{C})^{\otimes}$ the pullback

$$\begin{tikzcd}
\mathbf{LMod}^G_A(\mathscr{C})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \arrow[d] \\
\mathscr{O}^{\otimes} \arrow[r] & \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}
\end{tikzcd}$$

taken in $\mathbf{Op}_{G,\infty}$.

For $H \leq G$, this recovers a fiberwise left module construction

$$\mathbf{LMod}^G_A(\mathscr{C})^{\otimes}_H \simeq \mathbf{LMod}_{A_H}(\mathscr{C}_H)$$

by Theorem 2.2.4 (iii) whose restrictions are computed in the underlying G-∞-category of $\mathscr{C}^{\otimes}$ by another application of Theorem 2.2.4 (iii), so in particular, the underlying G-∞-category $\underline{\mathbf{LMod}}^G_A(\mathscr{C})^{\otimes}$ can be viewed as a G-parametrized version of Lurie's left module category.

Remark 3.3.4. There are already other parametrized treatments of module categories, for example by Linskens–Nardin–Pol [LNP25, Appendix A] or Pützstück [Pü25, Section 5]. Both of these references have global equivariant applications in mind and in particular focus on the fully commutative ('ultracommutative' or 'normed') setup, while our intention is to give a general $\mathscr{O}$-monoidal treatment in a non-global setup. However, the ideas in [LNP25] are quite close to our treatment.

We will now demonstrate that our construction endows $\mathbf{LMod}^G_A(\mathscr{C})$ with an $\mathscr{O}$-monoidal structure. Its proof depends on the following result generalizing [Ram22, Lemma 1.10] which is based on [HMS22, Lemma A.1.8]. We also took inspiration from [RZ25, Lemma B.6].

Lemma 3.3.5. Let $\mathcal{O} \in \mathbf{Op}_{G,\infty}$ with $\mathcal{C}^\otimes, \mathcal{D}^\otimes \in \mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_\infty)$ and let $F\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ be a map in $\mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_\infty)$. Suppose the following two conditions:

(i) For every $H \leq G$ and $o \in \mathcal{O}^\otimes_H$ the induced functor $F_o\colon \mathcal{C}^\otimes_o \to \mathcal{D}^\otimes_o$ is a coCartesian fibration.

(ii) Let $o \to o'$ be a map in $\mathcal{O}^\otimes$ such that o' lies over an orbit. Then, $\bigotimes_{o\to o'}\colon \mathcal{C}^\otimes_o \to \mathcal{C}^\otimes_{o'}$ sends F_o-coCartesian edges to $F_{o'}$-coCartesian edges.

Then, $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is a coCartesian fibration.

Proof. The criterion given in [HMS22, Lemma A.1.8] specialized to our setting states that the following conditions altogether imply that $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is a coCartesian fibration.

(1) The functors $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ and $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ are coCartesian fibrations.

(2) The functor $F\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ preserves coCartesian edges over $\mathcal{O}^\otimes$.

(3) For each $o \in \mathcal{O}^\otimes$ the induced map $F_o\colon \mathcal{C}^\otimes_o \to \mathcal{D}^\otimes_o$ is a coCartesian fibration.

(4) Let $o \to o'$ be a map in $\mathcal{O}^\otimes$. Then, the induced map $\mathcal{C}^\otimes_o \to \mathcal{C}^\otimes_{o'}$ takes F_o-coCartesian edges to $F_{o'}$-coCartesian edges.

Parts (1) and (2) are already contained in our assumptions. So let us confirm (3) and (4).

(3) Suppose $o \in \mathcal{O}^\otimes_X$ where we have an orbit decomposition $X \simeq \coprod_i G/H_i$ with $H_i \leq G$. In that regard, $\mathcal{O}^\otimes_X \simeq \prod_i \mathcal{O}^\otimes_{H_i}$ and o corresponds to some tuple $(o_i)_i$ under this correspondence. Then, the map becomes

$$F_o\colon \mathcal{C}^\otimes_o \simeq \prod_i \mathcal{C}^\otimes_{o_i} \xrightarrow{\prod_i F_{o_i}} \prod_i \mathcal{D}^\otimes_{o_i} \simeq \mathcal{D}^\otimes_o$$

which is a product of coCartesian fibrations by (i) and thus itself a coCartesian fibration.

(4) Suppose $o' \in \mathcal{O}^\otimes_Y$ with orbit decompositions $Y \simeq \coprod_j G/K_j$. Then, o' correspond to $(o'_j)_j$ and the map in question becomes

$$\mathcal{C}^\otimes_o \to \mathcal{C}^\otimes_{o'} \to \prod_j \mathcal{C}^\otimes_{o_j}.$$

So it suffices that the component maps $\mathcal{C}^\otimes_o \to \mathcal{C}^\otimes_{o_j}$ send F_o-coCartesian edges to F_{o_j}-coCartesian edges which is assumption (ii).

So we win. □

Corollary 3.3.6. Let $\mathcal{C}^\otimes, \mathcal{D}^\otimes \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_\infty)$ and consider a map $F\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ in $\mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_\infty)$. Suppose the following two conditions:

(i) For every $H \leq G$ the induced functor $F_H\colon \mathcal{C}^\otimes_H \to \mathcal{D}^\otimes_H$ is a coCartesian fibration.

(ii) Norms, fiberwise tensor products and restrictions preserve coCartesian edges, i.e. for $H \leq K \leq G$ the maps

$$\mathrm{Res}^K_H\colon \mathcal{C}^\otimes_K \to \mathcal{C}^\otimes_H,\ \otimes\colon \mathcal{C}^\otimes_H \times \mathcal{C}^\otimes_H \to \mathcal{C}^\otimes_H,\ \mathrm{N}^K_H\colon \mathcal{C}^\otimes_H \to \mathcal{C}^\otimes_K$$

preserve coCartesian edges over the respective fibers of $\mathcal{D}^\otimes$.

Then, $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is a coCartesian fibration.

Proof. We want to check the conditions in Lemma 3.3.5. Note here that $\bigoplus_{G/K \leftarrow G/H} = \mathrm{Res}^K_H$. An induction argument reduces allows us to only demand fiberwise tensor products of two objects. We furthermore note that the following are automatic:

- The forwards maps $\emptyset \to G/H$ induce units $\mathbb{1}\colon * \to \mathscr{C}^\otimes_H$ and this certainly preserves coCartesian edges since it sends id_* to $\mathrm{id}_{\mathbb{1}}$.
- The backwards maps $G/H \leftarrow \emptyset$ induce maps $\mathscr{C}^\otimes_H \to *$ which preserves coCartesian edges since every edge in $*$ is id_*-coCartesian.
- The backwards fold maps $G/H \leftarrow \coprod_i G/H$ induce diagonal maps $\mathscr{C}^\otimes_H \to \prod_i \mathscr{C}^\otimes_H$ which hence preserve coCartesian edges.

This is why there is no need to furthermore demand these in (ii). □

Theorem 3.3.7. Let $\mathscr{C}^\otimes \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_\infty)$ whose underlying G-∞-category has fiberwise geometric realizations and whose norms, fiberwise tensorings with one object and restrictions commute over these fiberwise geometric realizations. Let $\mathscr{O}^\otimes \in \mathbf{Op}_{G,\infty}$ and $A \in \mathbf{Alg}_{\mathscr{O}\otimes\mathbb{E}_1}(\mathscr{C})$. Then, $\mathbf{LMod}^G_A(\mathscr{C})^\otimes \to \mathscr{O}^\otimes$ is an $\mathscr{O}$-monoidal ∞-category.

Proof. It suffices to prove that $\mathbf{LMod}^G_A(\mathscr{C})^\otimes \to \mathscr{O}^\otimes$ is a coCartesian fibration. Since coCartesian fibrations pull back, it suffices to show that $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^\otimes \to \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^\otimes$ is a coCartesian fibration.

To do so we will check the criteria in Corollary 3.3.6.

(i) By Theorem 2.2.4 (i), for $H \leq G$ we have

$$\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^\otimes_H \simeq \mathbf{Alg}_{\mathcal{LM}}(\mathscr{C}_H) \to \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C}_H) \simeq \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^\otimes_H.$$

Using the assumption that fiberwise tensorings commute with fiberwise geometric realizations it follows that this is a coCartesian fibration from Lemma 3.3.1.

(ii) We now have to show that norms, restrictions and fiberwise tensor products preserve coCartesian edges. We do this via the explicit description of coCartesian edges (Lemma 3.3.1 (ii)).

- Fiberwise tensor products: Let $H \leq G$ and consider a coCartesian edge

$$((A_1, M_1), (A_2, M_2)) \to ((B_1, B_1 \otimes_{A_1} M_1), (B_2, B_2 \otimes_{A_2} M_2))$$

in $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^\otimes_H \times \mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^\otimes_H$ over $(A_1, A_2) \to (B_1, B_2)$ in $\mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^\otimes_H \times \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^\otimes_H$. We wish to show that

$$\begin{aligned}(A_1, M_1) \otimes (A_2, M_2) &\simeq (A_1 \otimes A_2, M_1 \otimes M_2) \\ &\to (B_1 \otimes B_2, (B_1 \otimes_{A_1} M_1) \otimes (B_2 \otimes_{A_2} M_2)) \\ &\simeq (B_1, B_1 \otimes_{A_1} M_1) \otimes (B_2, B_2 \otimes_{A_2} M_2)\end{aligned}$$

is a coCartesian edge over $A_1 \otimes A_2 \to B_1 \otimes B_2$. For this, we need that the natural map

$$(B_1 \otimes B_2) \otimes_{A_1 \otimes A_2} (M_1 \otimes M_2) \to (B_1 \otimes_{A_1} M_1) \otimes (B_2 \otimes_{A_2} M_2)$$

is an equivalence. This follows by writing out the bar construction and using (repeatedly) that tensoring with one object preserves geometric realizations and is symmetric monoidal.

- Norms: Let $H \leq G$ and consider a coCartesian edge
$$(A, M) \to (B, B \otimes_A M)$$
in $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^{\otimes}_{H}$ over $A \to B$ in $\mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}_{H}$. We wish to show that
$$(N_H^G A, N_H^G M) \simeq N_H^G(A, M) \to N_H^G(B, B \otimes_A M) \simeq (N_H^G B, N_H^G(B \otimes_A M))$$
is a coCartesian edge over $N_H^G A \to N_H^G B$. For this, we need that the natural map $N_H^G B \otimes_{N_H^G A} N_H^G M \to N_H^G(B \otimes_A M)$ is an equivalence. But that's true because N_H^G is symmetric monoidal (Lemma A.1.1) and commutes with geometric realizations by assumption.
- Restrictions: This is the exact same argument as for norms using Lemma A.1.1 again.

This finishes the proof. $\square$

Remark 3.3.8. The same strategy shows that $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \to \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}$ is a cartesian fibration.

Corollary 3.3.9. Let $\mathscr{C}^{\otimes} \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_{\infty})$ and $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$. Then, taking left modules enhances to a functor
$$\mathbf{LMod}^{G}_{(-)}(\mathscr{C})^{\otimes} \colon \mathbf{Alg}_{\mathscr{O}\otimes\mathbb{E}_1}(\mathscr{C}) \to \mathbf{Alg}_{\mathscr{O}}(\mathbf{Cat}_{\infty})$$
where the functoriality is through relative tensor products.

Proof. We have seen that $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \to \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}$ is a coCartesian fibration and symmetric monoidal (Theorem 3.3.7). By parametrized microcosmic monoidal straightening-unstraightening (Proposition 3.2.1) this corresponds to a lax G-symmetric monoidal functor $\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \to \underline{\mathbf{Cat}}^{\times}_{G,\infty}$. We win by applying $\mathbf{Alg}_{\mathscr{O}}(-)$. $\square$

Those conditions required in this theorem will be carried around in most of the subsequent results so that we have an $\mathscr{O}$-monoidal structure on $\mathbf{LMod}^{G}_{A}(\mathscr{C})$, so let us introduce a terminology for it.

Definition 3.3.10. Let $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$. An $\mathscr{O}$-module datum consists of a pair
$$\left(\mathscr{C}^{\otimes} \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_{\infty}),\ A \in \mathbf{Alg}_{\mathscr{O}\otimes\mathbb{E}_1}(\mathscr{C})\right)$$
such that the underlying G-∞-category of $\mathscr{C}^{\otimes}$ has fiberwise geometric realizations and whose norms, fiberwise tensorings with one object and restrictions commute over these fiberwise geometric realizations.

So the content of Theorem 3.3.7 is that an $\mathscr{O}$-module datum $(\mathscr{C}^{\otimes}, A)$ gives rise to an $\mathscr{O}$-monoidal ∞-category $\mathbf{LMod}^{G}_{A}(\mathscr{C})^{\otimes}$.

The device running the Thom spectrum engine later is operadic left Kan extensions (Theorem 2.3.13), which depends on a distributivity property (Definition 2.3.8), so we will spend the rest of the remaining section showing that the our left module category is $\mathscr{O}$-distributive. Since distributivity is treated in the setting of G-∞-categories, we will work with the underlying G-symmetric monoidal G-∞-categories in the rest of the section.

To talk about distributivity one already needs the existence of colimits, which is the main content of the following result. In particular, this needs a certain projection formula condition for which we recall that it is implied by distributivity (Proposition 2.3.12).

Proposition 3.3.11. Let $(\mathscr{C}^{\otimes}, A)$ be an $\mathbb{E}_0$-module datum.

(i) The forgetful functor $\underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}}) \to \underline{\mathscr{C}}$ is conservative.

(ii) Let $A \to B$ be a map of $\mathbb{E}_1$-algebras. Then, the levelwise relative tensor products assemble into a G-left adjoint $B \otimes_A -\colon \underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}}) \to \underline{\mathbf{LMod}}^G_B(\underline{\mathscr{C}})$.

(iii) Let $\underline{\mathscr{C}}$ be G-presentable and levelwise distributive. Assume moreover that projection formulas hold, i.e. for $H \le K \le G$ and objects $c \in \mathscr{C}_H, c' \in \mathscr{C}_K$ the preferred map

$$\mathrm{Ind}_H^K \left(\mathrm{Res}_H^K c' \otimes c \right) \to c' \otimes \mathrm{Ind}_H^K c$$

is an equivalence. Then, the G-∞-category $\underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}})$ is G-presentable and the G-functor $\underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}}) \to \underline{\mathscr{C}}$ strongly preserves and strongly reflects G-colimits.

Proof.

(i) Conservativity is a levelwise statement where it is [Lur17, Corollary 4.2.3.2].

(ii) The coCartesian fibration $\underline{\mathbf{Alg}}_{\mathcal{LM}}(\underline{\mathscr{C}}) \to \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\underline{\mathscr{C}})$ (see Theorem 3.3.7) straightens to a G-functor $\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\underline{\mathscr{C}}) \to \underline{\mathbf{Cat}}_{G,\infty}$ and is the straightened version of the **LMod** construction. So the map $A \to B$ yields a G-functor, which we denote by

$$B \otimes_A -\colon \underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}}) \to \underline{\mathbf{LMod}}^G_B(\underline{\mathscr{C}}).$$

Levelwise, this is the classical relative tensor products, i.e. by $B_H \otimes_{A_H} -$ on the level $H \le G$. So it is a relative adjunction over $\mathbf{Orb}_G^{\mathrm{op}}$ with levelwise right adjoint given by restriction, since this is levelwise so [Lur17, Proposition 7.3.2.6]. To check that it is a G-adjunction we still need to check that the right adjoints assemble into a G-functor [Hil24b, Corollary 2.2.7], but they do because they come from the cartesian straightening of the cartesian fibration $\underline{\mathbf{Alg}}_{\mathcal{LM}}(\underline{\mathscr{C}}) \to \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\underline{\mathscr{C}})$ (see Remark 3.3.8).

(iii) We begin with G-cocompleteness, which is equivalent to showing that $\underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}})$ is fiberwise cocomplete, that restrictions preserve fiberwise colimits and that restrictions admit left adjoints satisfying the Beck-Chevalley condition [Hil24b, Theorem 3.1.9]. We first note that the assumptions are enough to conclude that $\underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}})$ is fiberwise presentable [Lur17, Corollary 4.2.3.7(1)] and in particular fiberwise cocomplete.

Now let $H \le K \le G$, then $\mathbf{LMod}_{A_K}(\mathscr{C}_K) \to \mathbf{LMod}_{A_H}(\mathscr{C}_H)$ preserves colimits because the forgetful functor to $\mathscr{C}_K$ resp. $\mathscr{C}_H$ creates colimits [Lur17, Corollary 4.2.3.7(2)] but $\mathscr{C}_K \to \mathscr{C}_H$ preserves colimits by G-cocompleteness of $\underline{\mathscr{C}}$. Altogether, $\underline{\mathbf{LMod}}^G_A(\underline{\mathscr{C}})$ is G-presentable [Hil24b, Theorem 6.1.2].

The same argument shows that $\mathbf{LMod}_{A_K}(\mathscr{C}_K) \to \mathbf{LMod}_{A_H}(\mathscr{C}_H)$ also preserves limits. The adjoint functor theorem thus ensures a left adjoint $\mathrm{ind}_H^K\colon \mathbf{LMod}_{A_H}(\mathscr{C}_H) \to \mathbf{LMod}_{A_K}(\mathscr{C}_K)$. We will first show that it is computed underlying, i.e. that the square

$$\begin{tikzcd} \mathbf{LMod}_{A_H}(\mathscr{C}_H) \arrow[r, "\mathrm{ind}_H^K"] \arrow[d, "U"'] & \mathbf{LMod}_{A_K}(\mathscr{C}_K) \arrow[d, "U"] \\ \mathscr{C}_H \arrow[r, "\mathrm{Ind}_H^K"'] & \mathscr{C}_K \end{tikzcd}$$

commutes. Since $\mathbf{LMod}_{A_H}(\mathscr{C}_H)$ is generated by free A_H-modules under sifted colimits [Lur17, Proposition 4.7.3.14] and all functors in this square preserve colimits, it suffices to check this on free A_H-modules. So let $c \in \mathscr{C}_H$ and $A_H \otimes c \in \mathbf{LMod}_{A_H}(\mathscr{C}_H)$ be the free

A_H-module on c. Let $M \in \mathbf{LMod}_{A_K}(\mathscr{C}_K)$. For the sake of the following computation, let us introduce the notation $\mathrm{res}^K_H \colon \mathbf{LMod}_{A_K}(\mathscr{C}_K) \to \mathbf{LMod}_{A_H}(\mathscr{C}_H)$. Then, we compute

$$\begin{aligned}
\mathrm{Map}_{\mathbf{LMod}_{A_K}(\mathscr{C}_K)}(\mathrm{ind}^K_H(A_H \otimes_{\mathbb{1}} c), M) &\simeq \mathrm{Map}_{\mathbf{LMod}_{A_H}(\mathscr{C}_H)}(A_H \otimes_{\mathbb{1}} c, \mathrm{res}^K_H M) \\
&\simeq \mathrm{Map}_{\mathscr{C}_H}(c, U \,\mathrm{res}^K_H M) \\
&\simeq \mathrm{Map}_{\mathscr{C}_H}(c, \mathrm{Res}^K_H UM) \\
&\simeq \mathrm{Map}_{\mathscr{C}_K}(\mathrm{Ind}^K_H c, UM) \\
&\simeq \mathrm{Map}_{\mathbf{LMod}_{A_K}(\mathscr{C}_K)}(A_K \otimes_{\mathbb{1}} \mathrm{Ind}^K_H c, M).
\end{aligned}$$

Thus, we discover $\mathrm{ind}^K_H(A_H \otimes c) \simeq A_K \otimes \mathrm{Ind}^K_H c$. By the projection formula we may thus write down a chain of natural equivalences

$$\begin{aligned}
U \,\mathrm{ind}^K_H(A_H \otimes_{\mathbb{1}} c) &\simeq U(A_K \otimes_{\mathbb{1}} \mathrm{Ind}^K_H c) \\
&\simeq UA_K \otimes U \,\mathrm{Ind}^K_H c \\
&\simeq UA_K \otimes \mathrm{Ind}^H_K Uc \\
&\simeq (UA_H)_K \otimes \mathrm{Ind}^H_K Uc \\
&\simeq \mathrm{Ind}^K_H(UA_H \otimes Uc) \\
&\simeq \mathrm{Ind}^K_H U(A_H \otimes_{\mathbb{1}} c).
\end{aligned}$$

Since restrictions and inductions of left module categories are computed underlying, we deduce that the Beck-Chevalley maps are computed underlying where it is an equivalence because $\mathscr{C}$ is G-cocomplete. On the other hand, the forgetful functor is conservative by (i), so the Beck-Chevalley maps for $\underline{\mathbf{LMod}}^G_A$ are also equivalences.

To see that $\underline{\mathbf{LMod}}^G_A(\mathscr{C}) \to \underline{\mathscr{C}}$ strongly preserves G-colimits, we need to see that this is fiberwise the case and that induction is computed underlying. The fiberwise part is [Lur17, Corollary 4.2.3.7(2)] again and the induction part was the commutative diagram above.

Since the forgetful functor strongly preserves G-colimits and is conservative, it also strongly reflects G-colimits [Hil24a, Lemma 2.2.12]. □

Remark 3.3.12.

(i) We were not able to write down the indexed coproduct functor

$$\mathrm{ind}^K_H \colon \mathbf{LMod}_{A_H}(\mathscr{C}_H) \to \mathbf{LMod}_{A_K}(\mathscr{C}_K)$$

by hand and needed presentability to do so.

(ii) One might try to factor it through $\mathbf{LMod}_{\mathrm{Ind}^K_H A}(\mathscr{C}_K)$ by looking for natural maps induced by the adjunctions. However, this already fails because Ind^K_H is typically not lax symmetric monoidal (it would more naturally be oplax) and so $\mathrm{Ind}^K_H A$ does not even naturally obtain an algebra structure.

Informally, given some A_H-module M, we want an action of A_K on $\mathrm{Ind}^K_H M$. Inducing up our given action yields a map

$$\mathrm{Ind}^K_H(A_H \otimes M) \to \mathrm{Ind}^K_H M$$

which is not yet of the form $A_K \otimes \mathrm{Ind}^K_H M \to \mathrm{Ind}^K_H M$ but that's where the projection formula comes to the rescue, providing us with a map

$$A_H \otimes \mathrm{Ind}_H^K M \xrightarrow{\simeq} \mathrm{Ind}_H^K(A_H \otimes M) \longrightarrow \mathrm{Ind}_H^K M.$$

This is why we needed to assume the projection formula for this construction.

We now describe in full generality the indexed tensor product on left module categories.

Proposition 3.3.13. Let $\mathcal{O}^\otimes \in \mathbf{Op}_{G,\infty}$ have a single G-color and $(\mathscr{C}^\otimes, A)$ an $\mathcal{O}$-module datum. Consider a coCartesian lift $o' \to o$ in $\underline{\mathcal{O}}^\otimes$ of a map in $\underline{\mathbb{F}}_{G,*}$ over some $G/H \in \mathbf{Orb}_G$ corresponding to $f\colon U \to G/H$.

If $U \simeq \coprod_{i=1}^n G/H_i$ with $H_i \leq G$ is an orbit decomposition, then the map $\underline{\bigotimes}_{o'\to o}$ factors as

$$\begin{tikzcd} \underline{\mathbf{LMod}}^G_A(\mathscr{C})^\otimes_{\underline{o'}} \arrow[r] \arrow[d] & \underline{\mathbf{LMod}}^G_{\underline{\bigotimes}_{U\to G/H}\underline{A}^\otimes(o')}(\mathscr{C})_{\underline{H}} \arrow[r, "A\otimes_{\underline{\bigotimes}_{o'\to o}A}-"] \arrow[d] & \underline{\mathbf{LMod}}^G_A(\mathscr{C})^\otimes_{\underline{H}} \\ \underline{\mathscr{C}}^\otimes_{\underline{o'}} \arrow[r, "\underline{\bigotimes}_{o'\to o}"'] & \underline{\mathscr{C}}^\otimes_{\underline{o}} & \end{tikzcd}$$

where the left square commutes.

Proof Sketch. Since the proof of this result is notationally quite heavy, we will first sketch an argument of this result in the classical setting of symmetric monoidal ∞-categories for the convenience of the reader. In the actual proof we will then parametrize all ingredients that appear.

Let $\mathscr{C}^\otimes \to \mathbb{F}_*$ be a symmetric monoidal ∞-category and $A^\otimes \in \mathbf{Alg}_{\mathbb{E}_\infty}(\mathscr{C})$, then the special case for $\langle 2\rangle \to \langle 1\rangle$ is

$$\begin{tikzcd} \mathbf{LMod}_A(\mathscr{C}) \times \mathbf{LMod}_A(\mathscr{C}) \arrow[r] \arrow[d] & \mathbf{LMod}_{A\otimes A}(\mathscr{C}) \arrow[r, "A\otimes_{A\otimes A}-"] \arrow[d] & \mathbf{LMod}_A(\mathscr{C}) \\ \mathscr{C} \times \mathscr{C} \arrow[r, "\otimes"'] & \mathscr{C} & \end{tikzcd}$$

To obtain the top factorization we note that

$$\mathrm{St}\left(\mathbf{LMod}_A(\mathscr{C})^\otimes \to \mathbb{F}_*\right) \simeq \left(\mathbb{F}_* \xrightarrow{A^\otimes} \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^\otimes \to \mathbf{Cat}_\infty\right)$$

and the top composite is the effect of this functor on $\langle 2\rangle \to \langle 1\rangle$. This is first sent to $(A, A) \to A$ in $\mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^\otimes$ which we can factor through the coCartesian lift $(A, A) \to A \otimes A \to A$. That already recover the top line of the diagram. The second map induces a relative tensor product functor (Lemma 3.3.1) and we are left to check that the first map

$$\mathbf{LMod}_A(\mathscr{C}) \times \mathbf{LMod}_A(\mathscr{C}) \to \mathbf{LMod}_{A\otimes A}(\mathscr{C})$$

is compatible with the underlying tensor product. It suffices to check that

$$\begin{tikzcd} \mathbf{LMod}_A(\mathscr{C}) \times \mathbf{LMod}_A(\mathscr{C}) \arrow[r] \arrow[d] & \mathbf{LMod}_{A\otimes A}(\mathscr{C}) \arrow[d] \\ \mathbf{Alg}_{\mathcal{LM}}(\mathscr{C}) \times \mathbf{Alg}_{\mathcal{LM}}(\mathscr{C}) \arrow[r] & \mathbf{Alg}_{\mathcal{LM}}(\mathscr{C}) \end{tikzcd}$$

commutes because $\mathbf{Alg}_{\mathcal{LM}}(\mathscr{C})^\otimes \to \mathscr{C}^\otimes$ is symmetric monoidal. Naively, our idea is that the square should be the naturality square of some natural transformation between suitably chosen functors $\mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^\otimes \to \mathbf{Cat}_\infty$ evaluated at $(A, A) \to A \otimes A$. In reality, our choices won't give naturality squares for all maps in $\mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})^\otimes$ but there will be some partial naturality, in particular for those $(A, A) \to A$ that we need! We will check this via some coCartesian edge techniques. $\square$

Proof of Proposition 3.3.13. By applying $- \times_{\mathrm{Span}(\mathbb{F}_G)} \underline{\mathbb{F}}_{G,*}$ to the defining pullback square of $\underline{\mathbf{LMod}}^G_A(\mathscr{C})^{\otimes}$ we obtain the square

$$\begin{tikzcd} \underline{\mathbf{LMod}}^G_A(\mathscr{C})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \arrow[d, "q"] \\ \underline{\mathscr{O}}^{\otimes} \arrow[r, "\underline{A}^{\otimes}"'] & \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \end{tikzcd}$$

In particular,

$$\underline{\mathrm{St}}\left(\underline{\mathbf{LMod}}^G_A(\mathscr{C})^{\otimes} \to \underline{\mathscr{O}}^{\otimes}\right) \simeq \left(\underline{\mathscr{O}}^{\otimes} \xrightarrow{\underline{A}^{\otimes}} \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \xrightarrow{\underline{\mathrm{St}}(q)} \underline{\mathbf{Cat}}_{\mathcal{T},\infty}\right)$$

and the map $\underline{\bigotimes}_{o' \to o}$ that we want to understand is the image of $o' \to o$ under this functor, so

$$\underline{\bigotimes}_{o' \to o} \simeq \underline{\mathrm{St}}(q)(\underline{A}^{\otimes}(o') \to \underline{A}^{\otimes}(o)).$$

We factor $\underline{A}^{\otimes}(o') \to \underline{A}^{\otimes}(o)$ over a the lift of $U \to G/H$ along $\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \to \underline{\mathbb{F}}_{G,*}$ depicted as follows:

$$\begin{tikzcd} & \underline{\bigotimes}_{U \to G/H}\underline{A}^{\otimes}(o') \arrow[dr] & \\ \underline{A}^{\otimes}(o') \arrow[ur] \arrow[rr] & & \underline{A}^{\otimes}(o) \end{tikzcd} \qquad \begin{tikzcd} & G/H \arrow[dr, equal] & \\ U \arrow[ur] \arrow[rr] & & G/H \end{tikzcd}$$

By applying $\underline{\mathrm{St}}(q)$ on the left square the map $\underline{\bigotimes}_{o' \to o}$ factors as the composite

$$\underline{\mathbf{LMod}}^G_A(\mathscr{C})^{\otimes}_{\underline{o'}} \longrightarrow \underline{\mathbf{LMod}}^G_{\otimes_{U \to G/H}\underline{A}^{\otimes}(o')}(\mathscr{C})_{\underline{H}} \longrightarrow \underline{\mathbf{LMod}}^G_A(\mathscr{C})^{\otimes}_{\underline{o}}$$

by functoriality where the last term is also equivalent to the H-∞-category $\underline{\mathbf{LMod}}^G_{\underline{A}^{\otimes}(o)}(\mathscr{C})_{\underline{H}}$ since $\mathscr{O}^{\otimes}$ has a single G-color. In particular, the second map is the effect of functoriality of $\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C}) \to \underline{\mathbf{Cat}}_{G,\infty}$ on a map of algebras over the same $G/H \in \mathbf{Orb}_G^{\mathrm{op}}$ which is given by levelwise relative tensor products.

Now about the first map. Consider the following functors:

$$\begin{tikzcd} \underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \arrow[d, "q"'] & \\ \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \arrow[r, "\Phi"] \arrow[d, "p"'] & \underline{\mathbf{Cat}}_{G,\infty} \\ \underline{\mathbb{F}}_{G,\infty} \arrow[r, "\Psi"'] & \underline{\mathbf{Cat}}_{G,\infty} \end{tikzcd}$$

where Φ and Ψ are the respective parametrized straightenings, i.e. $\Phi = \underline{\mathrm{St}}(q)$ and $\Psi = \underline{\mathrm{St}}(pq)$. Ideally,[4] we would now like to demonstrate the existence of a G-natural transformation[5] $\Phi \Rightarrow \Psi p$ because naturality with respect to $\underline{A}^{\otimes}(o') \to \underline{\bigotimes}_{U \to G/H}\underline{A}^{\otimes}(o')$ then yields a commutative diagram

[4] In the end, we will not get a natural transformation but will still get some partial naturality.

[5] I.e. a map in the ∞-category $\mathrm{Fun}_G(\Phi, \Psi p)$. See [CLL23, Remark 2.2.3] for an unravelled version of this notion.

$$\begin{tikzcd}
\underline{\mathbf{LMod}}^G_{\underline{A}}(\mathscr{C})^{\otimes}_{\underline{o'}} \arrow[r] \arrow[d] & \underline{\mathbf{LMod}}^G_{\underline{\bigotimes}_{U\to t}\underline{A}^{\otimes}(o')}(\mathscr{C})_{\underline{H}} \arrow[d] \\
\underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes}_{\underline{U}} \arrow[r, "\underline{\bigotimes}_{U\to G/H}"'] & \underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes}_{\underline{H}}
\end{tikzcd}$$

and moreover there is a commutative diagram

$$\begin{tikzcd}
\underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes}_{\underline{U}} \arrow[r, "\underline{\bigotimes}_{U\to G/H}"] \arrow[d] & \underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes}_{\underline{H}} \arrow[d] \\
\underline{\mathscr{C}}^{\otimes}_{\underline{U}} \arrow[r, "\underline{\bigotimes}_{U\to G/H}"'] & \underline{\mathscr{C}}^{\otimes}_{\underline{H}}
\end{tikzcd}$$

because $\underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \to \mathscr{C}^{\otimes}$ is G-symmetric monoidal (Theorem 2.2.4). Pasting these two squares yields the desired factorization.

To find a natural transformation $\Phi \Rightarrow \Psi p$ it would be enough to give a map $\underline{\mathrm{Un}}(\Phi) \to \underline{\mathrm{Un}}(\Psi p)$ in $\mathrm{coCart}\big(\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}\big)$. By the behaviour of (parametrized) unstraightenings with compositions, we obtain a pullback square

$$\begin{tikzcd}
\underline{\mathrm{Un}}(\Psi p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Un}}(\Psi) \arrow[d] \\
\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \arrow[r, "p"'] & \underline{\mathbb{F}}_{\mathcal{T},*}
\end{tikzcd}$$

where we really use that we know this for the ordinary unstraightenings and reduce it to that case because the parametrized unstraightening of a functor to $\underline{\mathbf{Cat}}_{G,\infty}$ corresponds to the unstraightening of the corresponding functor to $\mathbf{Cat}_\infty$ (see Proposition 2.3.2). By definition, we have $\underline{\mathrm{Un}}(\Psi) \simeq \underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes}$, so

$$\underline{\mathrm{Un}}(\Psi p) \simeq \underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \times_{\underline{\mathbb{F}}_{\mathcal{T},*}} \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}.$$

Thus, we are considering the map

$$\begin{tikzcd}
\underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \arrow[rr, "(\mathrm{id},q)"] \arrow[dr, "q"'] & & \underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \times_{\underline{\mathbb{F}}_{G,*}} \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \arrow[dl, "\mathrm{pr}_2"] \\
& \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} &
\end{tikzcd}$$

which – as it turns out – is only a functor over $\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}$, but not a map of coCartesian fibrations, meaning that this does not induce a natural transformation $\Phi \Rightarrow \Psi p$. Nonetheless, we are still able to salvage some naturality squares out of this, namely exactly with respect to the maps $\underline{A}^{\otimes}(o') \to \underline{\bigotimes}_{U\to G/H}\underline{A}^{\otimes}(o')$ – which are precisely the ones we need.

We pull back this diagram along $\varphi\colon [1] \to \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}$ where we pick out any p-coCartesian edge in $\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}$. We claim now that

$$\begin{tikzcd}
\underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \times_{\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}} [1] \arrow[rr, "\widetilde{(\mathrm{id},q)}"] \arrow[dr, "\widetilde{q}"'] & & \underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \times_{\underline{\mathbb{F}}_{G,*}} \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \times_{\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes}} [1] \arrow[dl, "\widetilde{\mathrm{pr}}_2"] \\
& [1] &
\end{tikzcd}$$

is a map of coCartesian fibrations. Indeed, consider a $\widetilde{q}$-coCartesian edge and we wish to show that its image under $\widetilde{(\mathrm{id}, q)}$ is $\widetilde{\mathrm{pr}}_2$-coCartesian. Equivalently [Lur09, Proposition 2.4.1.3(1)], pick a q-coCartesian edge $x \to y$ over φ and we wish to show that $(x, qx) \to (y, qy)$ is pr_2-coCartesian. Note that pr_2 is the projection map of a pullback square

$$\begin{tikzcd}
\underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \times_{\underline{\mathbb{F}}_{G,*}} \underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \arrow[r] \arrow[d, "\mathrm{pr}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathbf{Alg}}_{\mathcal{LM}}(\mathscr{C})^{\otimes} \arrow[d, "pq"] \\
\underline{\mathbf{Alg}}_{\mathbb{E}_1}(\mathscr{C})^{\otimes} \arrow[r] & \underline{\mathbb{F}}_{G,*}
\end{tikzcd}$$

and being a coCartesian edge can be checked before pulling back [Lur09, Proposition 2.4.1.3(1)]. In other words, we are asking if $x \to y$ is pq-coCartesian. By assumption, $x \to y$ is q-coCartesian and moreover $q(x \to y) = \varphi$ is p-coCartesian. Thus, $x \to y$ is pq-coCartesian [Lur09, Proposition 2.4.1.3(3)].

This yields naturality for p-coCartesian edges and in particular gives the naturality for the commutative squares that we wanted. □

Now, we can finally prove distributivity of **LMod**. We are grateful to Kaif Hilman for suggesting the proof strategy.

Theorem 3.3.14. Let $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ have a single G-color and $(\mathscr{C}^{\otimes}, A)$ be an $\mathscr{O}$-module datum. Suppose that $\underline{\mathscr{C}}^{\otimes}$ is distributive and G-presentable. Then, $\underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes} \to \underline{\mathscr{O}}^{\otimes}$ is $\mathscr{O}$-distributive.

Proof. Let $H \leq G$ and $o \in \mathscr{O}^{\otimes}_{H}$. We then have $\underline{\mathbf{LMod}}^{\mathcal{T}}_{A}(\mathscr{C})^{\otimes}_{\underline{o}} \simeq \underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})_{\underline{H}}$ since $\mathscr{O}^{\otimes}$ has a single G-color (Lemma 2.1.3) and this is H-cocomplete by Proposition 3.3.11 (ii) and Proposition 2.3.12.

Now again let $H \leq G$ and $\alpha \colon o' \to o$ in $\underline{\mathscr{O}}^{\otimes}$ be a coCartesian lift of a map in $\underline{\mathbb{F}}_{G,*}$ over G/H corresponding to $f \colon U \to G/H$. We need to show that the associated pushforward H-functor $\underline{\bigotimes}_{o' \to o} \colon \underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{o'}} \to \underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{o}}$ is distributive. For this, consider any pullback square

$$\begin{tikzcd}
U' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & V' \arrow[d, "g"] \\
U \arrow[r, "f"'] & G/H
\end{tikzcd}$$

in $\mathbb{F}_G$ and $p \colon \underline{I}^{\triangleright} \to g'^{*}\underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{o'}}$ be a $(\mathbf{Orb}_G)_{/U'}$-colimit diagram. Let $U \simeq \coprod_i G/H_i$ with $H_i \leq G$ be an orbit decomposition, so $\underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{o'}} \simeq f_* \coprod_i \underline{\mathbf{LMod}}^{\mathcal{T}}_{A}(\mathscr{C})_{\underline{H_i}}$ by Theorem 2.3.6 and using that $\mathscr{O}^{\otimes}$ only has a single G-color. We need to prove that the composite

$$(f'_*\underline{I})^{\triangleright} \longrightarrow f'_*(\underline{I}^{\triangleright}) \longrightarrow f'_*g'^{*}\textstyle\coprod_i \underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{H_i}} \xrightarrow{\simeq} g^{*}f_*\coprod_i \underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{H_i}} \longrightarrow g^{*}\underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{o}}$$

is a colimit diagram. The last map in the above composite as

$$g^{*}f_*\underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{o'}} \longrightarrow g^{*}\underline{\mathbf{LMod}}^{G}_{\underline{\otimes}_{o'\to o}A(o')}(\mathscr{C})_{\underline{H}} \longrightarrow g^{*}\underline{\mathbf{LMod}}^{\mathcal{T}}_{A}(\mathscr{C})^{\otimes}_{\underline{H}}$$

by Proposition 3.3.13.

Then, there is a commutative diagram

$$\begin{tikzcd}
(f'_*\underline{I})^{\triangleright} \arrow[r] \arrow[d, equal] & f'_*(\underline{I}^{\triangleright}) \arrow[r] \arrow[d, equal] & f'_*g'^{*}\coprod_i \underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{H_i}} \arrow[r, "\simeq"] \arrow[d] & g^{*}\underline{\mathbf{LMod}}^{G}_{A}(\mathscr{C})^{\otimes}_{\underline{o'}} \arrow[r] \arrow[d] & g^{*}\underline{\mathbf{LMod}}^{G}_{\underline{\otimes}_{o'\to o}}(\mathscr{C})^{\otimes}_{\underline{H}} \arrow[d] \\
(f'_*\underline{I})^{\triangleright} \arrow[r] & f'_*(\underline{I}^{\triangleright}) \arrow[r] & f'_*g'^{*}\underline{\mathscr{C}}^{\otimes}_{\underline{o'}} \arrow[r, "\simeq"'] & g^{*}\underline{\mathscr{C}}^{\otimes}_{\underline{o'}} \arrow[r] & g^{*}\underline{\mathscr{C}}^{\otimes}_{\underline{H}}
\end{tikzcd}$$

induced by the forgetful functors by Proposition 3.3.13. Since the forgetful functor strongly preserves colimits (Proposition 3.3.11 (ii)) and by assumption $\underline{I}^{\triangleright} \to g'^*\underline{\mathbf{LMod}}_A^G(\mathscr{C})^{\otimes}_{\underline{\mathcal{O}'}}$ is a $(\mathbf{Orb}_G)_{/U}$-colimit diagram, we deduce that

$$\underline{I}^{\triangleright} \longrightarrow g'^*\underline{\mathbf{LMod}}_A^G(\mathscr{C})^{\otimes}_{\underline{\mathcal{O}'}} \longrightarrow \underline{\mathscr{C}}^{\otimes}_{\underline{\mathcal{O}'}}$$

is also a $(\mathbf{Orb}_G)_{/U'}$-colimit diagram, so since $\underline{\mathscr{C}}^{\otimes}$ is $\underline{\mathcal{O}}$-distributive, it follows that the bottom line of the above diagram is a colimit diagram. On the other hand, the forgetful functor $\underline{\mathbf{LMod}}_A^G(\mathscr{C})^{\otimes} \to \underline{\mathscr{C}}^{\otimes}$ strongly reflects colimits (Proposition 3.3.11), with which the upper line is also a colimit diagram.

To get back to our desired composite, we are only left to postcompose the top line by the map $g^*\underline{\mathbf{LMod}}^G_{\otimes_{i=1}^n N_{H_i}^H A_{H_i}}(\mathscr{C})^{\otimes}_{\underline{H}} \to g^*\underline{\mathbf{LMod}}_A^G(\mathscr{C})^{\otimes}_{\underline{H}}$, which is a $(\mathbf{Orb}_G)_{/V'}$-left adjoint (Proposition 3.3.11 (ii)) and thus strongly preserves $(\mathbf{Orb}_G)_{/V'}$-colimits. □

Let us enhance our terminology of $\mathcal{O}$-module data for the sake of brevity.

Definition 3.3.15. A distributive module datum consists of a triple $(\mathcal{O}^{\otimes}, \mathscr{C}^{\otimes}, A)$ such that $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ with a single G-color and $(\mathscr{C}^{\otimes}, A)$ is an $\mathcal{O}$-module datum (Definition 3.3.10) with distributive and G-presentable $\underline{\mathscr{C}}^{\otimes}$.

So the content of Theorem 3.3.14 is precisely that distributive module data yield distributive **LMod** constructions.

3.4 Parametrized Grouplike Spaces & Parametrized Picard Spaces

Grouplike monoidal G-spaces will be star players in this article. It's the language to phrase Picard spaces in which will be an ingredient in the definition of Thom spectra and it is also involved in the recognition theorem, which will allow us to run computational arguments. We will recall and set the foundations in this subsection.

We generalize [ABG18, Definition 7.1] and define:

Definition 3.4.1.

(i) An $\mathbb{E}_1$-G-space is grouplike if it is grouplike at every level.

(ii) Let $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $\eta\colon \mathrm{Infl}_G \mathbb{E}_1^{\otimes} \to \mathcal{O}^{\otimes}$ be a map in $\mathbf{Op}_{G,\infty}$. Let $X \in \mathbf{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_G)$, then X is grouplike (with respect to η) if $\eta^* X$ is a grouplike $\mathbb{E}_1$-G-space.

We write $\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathcal{O}}(\underline{\mathcal{S}}_G) \subseteq \underline{\mathbf{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_G)$ for the full subcategory of (levelwise) grouplike $\mathcal{O}$-monoidal spaces.[6]

Note that we are not yet talking about G-subcategories. We will verify that it forms a G-subcategory later (Lemma 3.4.6).

Remark 3.4.2. Let V be a finite-dimensional real G-representation. Juran considers a notion of grouplike algebras that allows more $\mathbb{E}_V$-operads than we are considering here [Jur25, Definition 2.6]. If $A \in \mathbf{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}_G)$, then $A^H \in \mathbf{Alg}_{\mathbb{E}_{\dim V^H}}(\mathcal{S})$ and Juran calls A grouplike if $\pi_0(A^H)$ is a group for all H such that $\dim V^H \geq 1$. So unlike us, he allows algebras which need not admit a levelwise monoid structure. On the other hand, we allow slightly more general G-∞-operads and not just the $\mathbb{E}_V$-operads.

[6] See [NS22, Definition 2.2.1] for a definition of parametrized algebra categories.

Classically, grouplike multiplicative spaces were studied by May with regards to his recognition theorem [May72]. Recently, such results were also proven in the equivariant setting, which we now briefly recall before moving to more categorical matters.

Theorem 3.4.3 ([CHLL24, Theorem A], [Jur25, Theorem A], [GM17b]).

(i) There is a G-symmetric monoidal equivalence of G-∞-categories $\underline{\mathbf{Sp}}^{\otimes}_{G,\geq 0} \simeq \underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathbb{E}^G_\infty}(\mathcal{S}^G)^{\otimes}$ which is given levelwise by Ω^∞.

(ii) Let V be a finite-dimensional real G-representation and $X, Y \in \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_V}(\underline{\mathcal{S}}_G)$ in the sense of Juran (Remark 3.4.2). Then, there exists a functor $\mathrm{B}^V\colon \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_V}(\underline{\mathcal{S}}_{G,*}) \to \mathcal{S}^G_*$ which induces an equivalence of G-spaces

$$\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathbb{E}_V}(\mathcal{S}^G)}(X, Y) \simeq \underline{\mathrm{Map}}_{\underline{\mathcal{S}}_{G,*}}(\mathrm{B}^V X, \mathrm{B}^V Y).$$

Lemma 3.4.4. Let $E \in \mathbf{Sp}^G_{\geq 0}$ and V be a G-representation. Then, $\mathrm{B}^V\Omega^\infty E \simeq \Omega^\infty\Sigma^V E$.

Proof. First, we note the adjunction

$$\begin{tikzcd} \mathbf{Sp}^G_{\geq 0} \arrow[r, shift left, hook] \arrow[r, phantom, "\perp"] & \mathbf{Sp}^G \arrow[l, shift left, "\tau_{\geq 0}"] \arrow[r, shift left, "\Sigma^V"] \arrow[r, phantom, "\perp"] & \mathbf{Sp}^G \arrow[l, shift left, "\Omega^V"] \end{tikzcd}$$

which restricts to an adjunction $\Sigma^V \dashv \tau_{\geq 0}\Omega^V$ on $\mathbf{Sp}^G_{\geq 0}$. Equipped with this, we perform a Yoneda argument: Let $X \in \mathbf{Sp}^G_{\geq 0}$, then

$$\begin{aligned} \mathrm{Map}_{\mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}^G_\infty}(\underline{\mathcal{S}}_G)}(\mathrm{B}^V\Omega^\infty E, \Omega^\infty X) &\simeq \mathrm{Map}_{\mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}^G_\infty}(\underline{\mathcal{S}}_G)}(\Omega^\infty E, \Omega^V\Omega^\infty X) \\ &\simeq \mathrm{Map}_{\mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}^G_\infty}(\underline{\mathcal{S}}_G)}(\Omega^\infty E, \Omega^\infty\Omega^V X) \\ &\simeq \mathrm{Map}_{\mathbf{Sp}^G_{\geq 0}}(E, \tau_{\geq 0}\Omega^V X) \\ &\simeq \mathrm{Map}_{\mathbf{Sp}^G_{\geq 0}}(\Sigma^V E, X) \\ &\simeq \mathrm{Map}_{\mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}^G_\infty}(\underline{\mathcal{S}}_G)}(\Omega^\infty\Sigma^V E, \Omega^\infty X) \end{aligned}$$

where we use the recognition theorem (Theorem 3.4.3) to argue that $\Omega^\infty X$ for $X \in \mathbf{Sp}^G_{\geq 0}$ hits all of $\mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}^G_\infty}(\underline{\mathcal{S}}_G)$. $\square$

Example 3.4.5. By (Real) equivariant Bott periodicity [Ati66] we have

$$\Omega^\infty \mathrm{ku}_{\mathbb{R}} \simeq \mathbb{Z} \times \mathrm{BU}_{\mathbb{R}}, \quad \Omega^\infty\Sigma^\rho \mathrm{ku}_{\mathbb{R}} \simeq \mathrm{BU}_{\mathbb{R}}, \quad \Omega^\infty\Sigma^{2\rho} \mathrm{ku}_{\mathbb{R}} \simeq \mathrm{BSU}_{\mathbb{R}},$$

and hence $\mathrm{B}^\rho\mathrm{BU}_{\mathbb{R}} \simeq \mathrm{BSU}_{\mathbb{R}}$.

Let us now move to categorical matters.

Lemma 3.4.6. The subcategory $\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathcal{O}}(\underline{\mathcal{S}}_G) \subseteq \underline{\mathbf{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_G)$ is a G-subcategory.

Proof. On the level $H \leq G$ the G-∞-category $\underline{\mathbf{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_G)$ is given by $\mathbf{Alg}_{\underline{\mathcal{O}_H}}(\underline{\mathcal{S}}_H)$ [NS22, Definition 2.2.1] [CLL23, Corollary 2.2.11(3)]. We must check that the restriction functor preserves grouplike algebras. For this, we observe for $H \leq K \leq G$ that the square

$$\begin{tikzcd} \mathbf{Alg}_{\underline{\mathcal{O}_K}}(\underline{\mathcal{S}}_K) \arrow[r] \arrow[d, "\eta^*"'] & \mathbf{Alg}_{\underline{\mathcal{O}_H}}(\underline{\mathcal{S}}_H) \arrow[d, "\eta^*"] \\ \mathbf{Alg}_{\mathrm{Infl}_K \mathbb{E}_1}(\underline{\mathcal{S}}_K) \arrow[r] & \mathbf{Alg}_{\mathrm{Infl}_H \mathbb{E}_1}(\underline{\mathcal{S}}_H) \end{tikzcd}$$

commutes, at least evaluated on objects. Indeed, if $A\colon \underline{\mathscr{O}}_K^{\otimes} \to \underline{\mathcal{S}}_K^{\times}$ is an object in the top left, then going around the square is given by the two equivalent formulations

$$\left(\mathrm{Infl}_K \mathbb{E}_1|_{\mathbf{Orb}_H^{\mathrm{op}}} \to \underline{\mathscr{O}}_K^{\otimes}|_{\mathbf{Orb}_H^{\mathrm{op}}} \to \underline{\mathcal{S}}_K|_{\mathbf{Orb}_H^{\mathrm{op}}}\right) \simeq \left(\mathrm{Infl}_K \mathbb{E}_1 \to \underline{\mathscr{O}}_K^{\otimes} \to \underline{\mathcal{S}}_K\right)|_{\mathbf{Orb}_H^{\mathrm{op}}}.$$

On the other hand, the restriction of a grouplike algebra is grouplike again since being grouplike is a levelwise condition. □

Lemma 3.4.7. The G-∞-category $\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)$ is G-presentable.

Proof. We need to show that it is G-cocomplete and fiberwise presentable [Hil24b, Theorem 6.1.3(7)].

- G-cocompleteness: Since $\underline{\mathcal{S}}_G^{\times}$ is distributive, we deduce that $\underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)$ is G-cocomplete [NS22, Theorem 5.1.4]. We show that G-colimits of grouplike algebras in $\underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)$ are grouplike again.

  On the fiber $H \leq G$ we are looking at $\mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}_H) \subseteq \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{S}_H)$ by **2.2.4** (iii). The fiberwise colimit statement is [Lur17, Remark 5.2.6.9], so we still need to discuss indexed coproducts. For $H \leq K \leq G$ this is the left adjoint of $\mathrm{Res}_H^K\colon \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{S}_K) \to \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{S}_H)$, also known as Ind_H^K. Since the symmetric monoidal structure on $\mathcal{S}_H$ resp. $\mathcal{S}_K$ is given by the pointwise cartesian symmetric monoidal structure, this is equivalently the functor

  $$\mathrm{Res}_H^K\colon \mathrm{Fun}\left(\mathbf{Orb}_K^{\mathrm{op}}, \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{S})\right) \to \mathrm{Fun}\left(\mathbf{Orb}_H^{\mathrm{op}}, \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{S})\right)$$

  induced by precomposition by $j\colon \mathbf{Orb}_H^{\mathrm{op}} \to \mathbf{Orb}_K^{\mathrm{op}}$. So its left adjoint Ind_H^K is given by left Kan extension. Let $A \in \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}_H) \simeq \mathrm{Fun}\left(\mathbf{Orb}_H^{\mathrm{op}}, \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S})\right)$ and $L \leq K$. Using the left Kan extension formula we compute

  $$(\mathrm{Ind}_H^K A)_L \simeq \mathrm{colim}\left(j_{\downarrow L} \to \mathbf{Orb}_K^{\mathrm{op}} \xrightarrow{A} \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{S})\right),$$

  which is a colimit of grouplike $\mathbb{E}_1$-spaces and thus itself grouplike [Lur17, Remark 5.2.6.9]. Lastly, $\mathrm{Res}_H^K\colon \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}_K) \to \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}_H)$ preserves colimits because colimits of functor categories are computed pointwise.

  Altogether, we have shown G-cocompleteness [Hil24b, Theorem 3.1.9].

- Fiberwise presentability: Since $\underline{\mathcal{S}}_G$ is distributive and fiberwise presentable, we may deduce that $\underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)$ is also fiberwise presentable [NS22, Theorem 5.1.4]. Moreover, the inclusion

  $$\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G) \hookrightarrow \underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)$$

  fiberwise preserves colimits and limits because it classically levelwise admits left and right adjoints. Thus, by the ∞-categorical reflection principle [RS22, Theorem 6.2], $\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)$ is also fiberwise presentable.

We're done. □

Proposition 3.4.8. The inclusion

$$\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G) \hookrightarrow \underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G).$$

is a G-symmetric monoidal G-left adjoint.

Proof. In the proof of the previous result (Lemma 3.4.7) we have seen that this inclusion functor is a G-cocontinuous. By G-presentability (Lemma 3.4.7) we can invoke the adjoint functor theorem [Hil24b, Theorem 6.2.1] to obtain a right adjoint. We still need to check that the inclusion is G-symmetric monoidal.

Fiberwise this follows because π_0 commutes with finite products. We need to discuss indexed products, so let $H \leq K \leq G$ and $X \in \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S}_H) \simeq \mathrm{Fun}\left(\mathbf{Orb}_H^{\mathrm{op}}, \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S})\right)$ and we wish to show that $\mathrm{Coind}_H^K X$ is grouplike as well. Let $L \leq K$. We compute

$$\begin{aligned}\pi_0^L \mathrm{Coind}_H^K X &\cong \pi_0 \mathrm{Map}_{\mathcal{S}}\left(*, (\mathrm{Coind}_H^K X)^L\right)\\ &\cong \pi_0 \mathrm{Map}_{\mathcal{S}_L}(*, \mathrm{Res}_L^K \mathrm{Coind}_H^K X)\\ &\cong \pi_0 \mathrm{Map}_{\mathcal{S}_K}(\mathrm{Ind}_L^K *, \mathrm{Coind}_H^K X)\\ &\cong \pi_0 \mathrm{Map}_{\mathcal{S}_H}(\mathrm{Res}_H^K \mathrm{Ind}_L^K *, X)\end{aligned}$$

By the double coset formula, $\mathrm{Res}_H^K \mathrm{Ind}_L^K *$ is a finite coproduct of orbits, so the above term is a finite product of equivariant π_0's of X, which is grouplike. □

Remark 3.4.9. While [ABG18, Lemma 7.2] is stated for a general (coherent) ∞-operad $\mathscr{O}$, the proof a priori only works for $\mathbb{E}_k$-operads. They cite [Lur12, Remark 5.1.3.5][7] only works for $\mathbb{E}_k$-operads. Moreover, they state that products of invertible objects remain invertible, but it is not clear why this should be true since the operations induced by $\mathscr{O}$ need not be compatible with the $\mathbb{E}_1$-operations that are picked out. Nonetheless, this argument still works for $\mathbb{E}_n$-operads by working slightly harder.

Definition 3.4.10. We will denote the G-right adjoint by $\mathrm{GL}_1 \colon \underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G) \to \underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)$.

Remark 3.4.11. Since GL_1 is fiberwise, say for $H \leq G$, a right adjoint

$$\mathrm{Fun}\left(\mathbf{Orb}_H^{\mathrm{op}}, \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{S})\right) \to \mathrm{Fun}\left(\mathbf{Orb}_H^{\mathrm{op}}, \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S})\right),$$

it is induced by the classical $\mathrm{GL}_1 \colon \mathbf{Alg}_{\mathbb{E}_1}(\mathcal{S}) \to \mathbf{Alg}^{\mathrm{gp}}_{\mathbb{E}_1}(\mathcal{S})$ [ABG18, Lemma 7.2]. As such, it fiberwise takes the subalgebra of tensor-invertible objects.

Corollary 3.4.12. Let $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$. Then, the functor GL_1 induces a G-right adjoint

$$\mathrm{GL}_1 \colon \underline{\mathbf{Alg}}_{\mathscr{O}\otimes \mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G) \to \underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathscr{O}\otimes \mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)$$

where being grouplike is with respect to the map $\mathrm{Infl}_G \mathbb{E}_1 \to \mathscr{O}^{\otimes} \otimes \mathrm{Infl}_G \mathbb{E}_1$ induced by the unit of the Stewart's Boardman–Vogt monoidal structure.

Proof. By Proposition 3.4.8 the left adjoint of $\mathrm{GL}_1 \colon \underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G) \to \underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)$ is G-symmetric monoidal, so GL_1 inherits a G-lax symmetric monoidal structure [Ste25b, Corollary D.5]. In particular, we can apply $\mathbf{Alg}_{\mathscr{O}}(-)$ and we conclude by the universal property of Stewart's Boardman–Vogt tensor product (Theorem 2.2.4 (i)) to obtain a G-right adjoint

$$\mathrm{GL}_1 \colon \underline{\mathbf{Alg}}_{\mathscr{O}\otimes \mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G) \to \underline{\mathbf{Alg}}_{\mathscr{O}}\left(\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)\right).$$

Let us still identify $\mathbf{Alg}_{\mathscr{O}}\left(\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)\right)$. Consider the commutative diagram

[7] This is [Lur17, Remark 5.2.6.9].

$$\begin{tikzcd}
\mathbf{Alg}_{\mathscr{O}}\left(\underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)\right) \arrow[r, "\simeq"] & \mathbf{Alg}_{\mathscr{O}\otimes \mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G) \\
\mathbf{Alg}_{\mathrm{Infl}_G \mathbb{E}_0}\left(\underline{\mathbf{Alg}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)\right) \arrow[u] \arrow[r, "\simeq"'] & \mathbf{Alg}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G) \arrow[u]
\end{tikzcd}$$

Then, $\mathbf{Alg}_{\mathscr{O}}\left(\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)\right)$ correspond to those objects in the top left corner of the square such that pulling back along the left vertical map yields a grouplike object. By commutativity of the diagram this corresponds to an object in the top right corner of the square such that pulling back along the right vertical map yields a grouplike object. In other words,

$$\mathbf{Alg}_{\mathscr{O}}\left(\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G)\right) \simeq \mathbf{Alg}^{\mathrm{gp}}_{\mathscr{O}\otimes \mathrm{Infl}_G \mathbb{E}_1}(\underline{\mathcal{S}}_G),$$

so we win. □

Remark 3.4.13.

(i) This result is already new for the classical case $G = *$ and ∞-operads $\mathscr{O}$ such that $\mathscr{O} \otimes \mathbb{E}_1$ is not an $\mathbb{E}_n$-operad.

(ii) On the other hand, Juran also discusses GL_1 for his notion of grouplike G-spaces [Jur25, Proposition 4.1] and $\mathbb{E}_V$-operads.

(iii) If $R \in \mathbf{Alg}_{\mathbb{E}_\infty^G}(\underline{\mathcal{S}}_G)$, then $\mathrm{GL}_1\, R$ deloops to $\mathrm{gl}_1 R \in \mathbf{Sp}_{G,\geq 0}$ by the recognition theorem (Theorem 3.4.3).

Here is a folklore computation of the homotopy groups of $\mathrm{GL}_1\, R$.

Lemma 3.4.14. Let $R \in \mathbf{Alg}_{\mathbb{E}_1}(\underline{\mathcal{S}}_G)$ and let V be a G-representation with $V^G \neq 0$. Then,

$$\pi_0^G(\mathrm{GL}_1(R)) \cong \pi_0^G(R)^\times \quad \text{and} \quad \pi_V^G(\mathrm{GL}_1(R)) \cong \pi_V^G(R).$$

Proof. Consider the pullback square of G-spaces

$$\begin{tikzcd}
\mathrm{GL}_1(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Omega^\infty R \arrow[d] \\
\underline{\pi}_0(R)^\times \arrow[r] & \underline{\pi}_0(R).
\end{tikzcd}$$

In particular, $\mathrm{GL}_1(R)^G$ consists of the the connected components of $(\Omega^\infty R)^G$ given by $\pi_0^G(R)^\times$. This yields the first part of the statement.

For the other part we need to argue that

$$\pi_V^G(\mathrm{GL}_1\, R, 1) \cong \pi_V^G(\Omega^\infty R, 1) \cong \pi_V^G(R).$$

Indeed, the elements in the first group can be written as a compatible system of based continuous maps $S^{V^H} \to (\mathrm{GL}_1\, R)^H$ for all $H \leq G$ up to based homotopy. Since $V^G \neq 0$, we infer $V^H \neq 0$ for all $H \leq G$, so S^{V^H} is connected. On the other hand, $(\mathrm{GL}_1\, R)^H$ is a union of connected components of $(\Omega^\infty R)^H$. Since we are considering based maps with the same basepoint in the target, we deduce that the groups must be the same. □

Lemma 3.4.15. Let $\mathscr{O}^\otimes \in \mathbf{Op}_{G,\infty}$ and $\mathscr{C}^\otimes \in \mathbf{Mon}_{\mathscr{O}}(\mathbf{Cat}_\infty)$. Then, the $\mathscr{O}$-monoidal structure of $\mathscr{C}^\otimes$ restricts onto an $\underline{\mathscr{O}}$-monoidal structure on $\underline{\mathscr{C}}^{\mathrm{core}}$.

Proof. The functor $(-)^{\mathrm{core}}\colon \mathbf{Cat}_\infty \to \mathcal{S}$ induces a G-right adjoint $\underline{(-)}^{\mathrm{core}}\colon \underline{\mathbf{Cat}}_{\infty,G} \to \underline{\mathcal{S}}_G$ on the corresponding cofree G-∞-categories [CLL23, Example 2.3.3]. Thus, it preserves parametrized products and hence corresponds to a G-symmetric monoidal functor $\underline{\mathbf{Cat}}^{\times}_{\infty,G} \to \underline{\mathcal{S}}^{\times}_G$ by [Ste25b, Corollary A.21]. This induces a functor

$$\underline{(-)}^{\mathrm{core}}\colon \mathbf{Alg}_{\underline{\mathcal{O}}}\left(\underline{\mathbf{Cat}}^{\times}_{\infty,G}\right) \to \mathbf{Alg}_{\underline{\mathcal{O}}}(\underline{\mathcal{S}}^{\times}_G)$$

which endows $\underline{\mathscr{C}}^{\mathrm{core}}$ with an $\underline{\mathcal{O}}$-monoidal structure. ☐

Construction 3.4.16. Let $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $\mathscr{C}^{\otimes} \in \mathbf{Mon}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{Cat}_\infty)$.

(i) We denote by $\underline{\mathrm{Pic}}_G(\mathscr{C}) = \mathrm{GL}_1(\underline{\mathscr{C}}^{\mathrm{core}})$ the Picard G-space of $\mathscr{C}$.

In particular, it is a G-space and enhances to an $(\mathcal{O}\otimes\mathbb{E}_1)$-submonoidal G-∞-category $\underline{\mathrm{Pic}}^{\otimes}_G(\mathscr{C}) \subseteq \underline{\mathscr{C}}^{\otimes}$ by the hard work of this subsection (Corollary 3.4.12, Lemma 3.4.15).

(ii) Let $\mathscr{P}^{\otimes} \to \mathcal{O}^{\otimes} \otimes \mathbb{E}_1^{\otimes}$ be a map in $\mathbf{Op}_{G,\infty}$ and $A \in \mathbf{Alg}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathscr{C})$. We define $\mathrm{Pic}^{\otimes}_G(\mathscr{C})_{\downarrow A}$ as the pullback

$$\begin{tikzcd} \underline{\mathrm{Pic}}^{\otimes}_G(\mathscr{C})_{\downarrow A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathscr{C}}^{\otimes}_{/A} \arrow[d] \\ \underline{\mathrm{Pic}}^{\otimes}_G(\mathscr{C}) \arrow[r] & \underline{\mathscr{C}}^{\otimes} \end{tikzcd}$$

in $\mathbf{Mon}^{\mathrm{NS}}_{\mathcal{O}\otimes\mathbb{E}_1}(\mathbf{Cat}_\infty)$.

(iii) Let $(\mathscr{C}^{\otimes}, A)$ be an $(\mathcal{O}\otimes\mathbb{E}_1)$-module datum. We write $\underline{\mathrm{Pic}}^{\otimes}_G(A) = \underline{\mathrm{Pic}}^{\otimes}_G(\mathbf{LMod}_A(\mathscr{C}))$ and likewise for similar variants.

If we wish to use the Barkan–Haugseng–Steinebrunner formalism of equivariant higher algebra, then we will omit the underlines.

Part (ii) is merely an example of a (monoidal enhancement) of a comma ∞-category but we single out this specimen because it takes a crucial role in our subsequent discussions of Thom spectra. Since this discussion will often involve left module categories, we shorten the notation in (iii), but we note that $\mathscr{C}$ is implicit in the notation.

Remark 3.4.17. Pützstück has also constructed monoidal structures on parametrized Picard spaces, most notably on versions appearing in global equivariant homotopy theory [Pü25]. He focuses on the ultracommutative setting while we discuss general G-∞-operads.

Lemma 3.4.18. Let $\mathcal{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $(\mathscr{C}^{\otimes}, A)$ be an $(\mathcal{O}\otimes\mathbb{E}_1)$-module datum and consider a map $\mathscr{P}^{\otimes} \to \mathcal{O}^{\otimes} \otimes \mathbb{E}_1^{\otimes}$ in $\mathbf{Op}_{G,\infty}$. Let $R \to A$ be a map in $\mathbf{Alg}_{\mathscr{P}\otimes\mathbb{E}_1}(\mathscr{C})$. Then, there is a pullback square

$$\begin{tikzcd} \underline{\mathrm{Pic}}^{\otimes}_G(R)_{\downarrow A} \arrow[r, "\mathrm{Ind}_R^A"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Pic}}^{\otimes}_G(A)_{\downarrow A} \arrow[d] \\ \underline{\mathrm{Pic}}^{\otimes}_G(R) \arrow[r, "\mathrm{Ind}_R^A"'] & \underline{\mathrm{Pic}}^{\otimes}_G(A) \end{tikzcd}$$

in $\mathbf{Mon}^{\mathrm{NS}}_{\mathscr{P}}(\mathbf{Cat}_\infty)$.

Proof. Note that Ind_R^A is $\mathscr{P}$-monoidal (Corollary 3.3.9). The commutativity of the square defines a comparison map

$$\underline{\mathrm{Pic}}_G^{\otimes}(R)_{\downarrow A} \to \underline{\mathrm{Pic}}_G^{\otimes}(A)_{\downarrow A} \times_{\underline{\mathrm{Pic}}_G^{\otimes}(A)} \underline{\mathrm{Pic}}_G^{\otimes}(R)$$

in $\mathbf{Mon}_{\mathscr{P}}^{\mathrm{NS}}(\mathbf{Cat}_\infty)$. It thus suffices to check that this is an equivalence on underlying ∞-categories. Consider the commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Pic}}_G(R)_{\downarrow A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathbf{LMod}}^G_{R/A} \arrow[r, "\mathrm{Ind}_R^A"] \arrow[d] & \underline{\mathbf{LMod}}^G_{A/A} \arrow[d] \\
\underline{\mathrm{Pic}}_G(R) \arrow[r] & \underline{\mathbf{LMod}}^G_R \arrow[r, "\mathrm{Ind}_R^A"'] & \underline{\mathbf{LMod}}^G_A
\end{tikzcd}$$

The right square is a pullback square by a parametrized version of a general category theory result: If $L\colon \mathscr{C} \leftrightarrows \mathscr{D}\colon R$ is an adjunction, then the composite rectangle

$$\begin{tikzcd}
\mathscr{C}_{/Rd} \arrow[r, "L"] \arrow[d] & \mathscr{D}_{/LRd} \arrow[r, "\epsilon_d"] \arrow[d] & \mathscr{D}_{/d} \arrow[d] \\
\mathscr{C} \arrow[r] & \mathscr{D} \arrow[r, equal] & \mathscr{D}
\end{tikzcd}$$

is a pullback square.[8]

Thus, we obtain

$$\begin{aligned}
\underline{\mathrm{Pic}}_G(R)_{\downarrow A} &\simeq \underline{\mathbf{LMod}}^G_{A/A} \times_{\underline{\mathbf{LMod}}^G_A} \underline{\mathrm{Pic}}_G(R) \\
&\simeq \underline{\mathbf{LMod}}^G_{A/A} \times_{\underline{\mathbf{LMod}}^G_A} \underline{\mathrm{Pic}}_G(A) \times_{\underline{\mathrm{Pic}}_G(A)} \underline{\mathrm{Pic}}_G(R) \\
&\simeq \underline{\mathrm{Pic}}_G(A)_{\downarrow A} \times_{\underline{\mathrm{Pic}}_G(A)} \underline{\mathrm{Pic}}_G(R)
\end{aligned}$$

as desired. □

Corollary 3.4.19. Let $\mathscr{O}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $(\mathscr{C}^{\otimes}, A)$ be an $(\mathscr{O} \otimes \mathbb{E}_1)$-module datum and consider a map $\mathscr{P}^{\otimes} \to \mathscr{O}^{\otimes} \otimes \mathbb{E}_1^{\otimes}$ in $\mathbf{Op}_{G,\infty}$. Let $R \to A$ be a map in $\mathbf{Alg}_{\mathscr{P} \otimes \mathbb{E}_1}(\mathscr{C})$. Then,

$$\mathrm{GL}_1(\underline{\mathrm{Pic}}_G^{\otimes}(R)_{\downarrow A}) \simeq \underline{\mathrm{Pic}}_G^{\otimes}(R) \times_{\underline{\mathrm{Pic}}_G^{\otimes}(A)} *$$

in $\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})$.

Proof. The functor GL_1 is a Bousfield colocalization, so applying it to the pullback in the previous Lemma (Lemma 3.4.18) yields

$$\mathrm{GL}_1(\underline{\mathrm{Pic}}_G^{\otimes}(R)_{\downarrow A}) \simeq \underline{\mathrm{Pic}}_G^{\otimes}(R) \times_{\underline{\mathrm{Pic}}_G^{\otimes}(A)} \mathrm{GL}_1(\underline{\mathrm{Pic}}_G^{\otimes}(A)_{\downarrow A}).$$

Since $\underline{\mathrm{Pic}}_G^{\otimes}(A)_{\downarrow A}$ is already a space, we have $\mathrm{GL}_1(\underline{\mathrm{Pic}}_G^{\otimes}(A)_{\downarrow A}) \simeq \mathrm{GL}_1\left((\underline{\mathrm{Pic}}_G^{\otimes}(A)_{\downarrow A})^{\mathrm{core}}\right)$ and applying $\mathrm{GL}_1 \circ (-)^{\mathrm{core}}$ to the defining pullback square of $\underline{\mathrm{Pic}}_G^{\otimes}(A)_{\downarrow A}$ yields a pullback square

$$\begin{tikzcd}
\mathrm{GL}_1(\underline{\mathrm{Pic}}_G^{\otimes}(A)_{\downarrow A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{GL}_1\,\mathrm{core}(\underline{\mathbf{LMod}}^G_A(\mathscr{C})^{\otimes}_{/A}) \arrow[d] \\
\mathrm{GL}_1(\underline{\mathrm{Pic}}_G^{\otimes}(A)) \arrow[r] & \mathrm{GL}_1\,\mathrm{core}(\underline{\mathbf{LMod}}^G_A(\mathscr{C})^{\otimes})
\end{tikzcd}$$

[8]Here are two ways of checking this:

1. Check that the comparison map $\mathscr{C}_{/Rd} \to \mathscr{D}_{/d} \times_{\mathscr{D}} \mathscr{C}$ is essentially surjective and fully faithful.
2. Unstraighten both sides of $\mathrm{Map}_{\mathscr{C}}(-, Rd) \simeq \mathrm{Map}_{\mathscr{D}}(L-, d)$.

which unravels to

$$\begin{tikzcd} \mathrm{GL}_1(\underline{\mathrm{Pic}}^{\otimes}_G(A)_{\downarrow A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \underline{\mathrm{Pic}}^{\otimes}_G(A)_{/A} \arrow[d] \\ \underline{\mathrm{Pic}}^{\otimes}_G(A) \arrow[r] & \underline{\mathrm{Pic}}^{\otimes}_G(A) \end{tikzcd}$$

so $\mathrm{GL}_1(\underline{\mathrm{Pic}}^{\otimes}_G(A)_{\downarrow A}) \simeq \underline{\mathrm{Pic}}^{\otimes}_G(A)_{/A} \simeq *$. □

4 Multiplicative Parametrized Thom Spectra

4.1 Setup & Universal Property of Multiplicative Parametrized Thom Spectra

We have finally assembled all ingredients to equivariantize the universal property of multiplicative Thom spectra from Antolín-Camarena–Barthel [ACB19]. Once the language is set up, our proofs are essentially the same as in [ACB19, Section 3.1], but we will write them out for the convenience of the reader (Theorem 4.1.8, Corollary 4.2.7). Let us first recall a parametrized definition of (multiplicative) equivariant Thom spectra and then prove a parametrized version of the Antolín-Camarena–Barthel universal property.

A parametrized version of the Ando–Blumberg–Gepner–Hopkins–Rezk [ABG+14] Thom spectrum definition gives:

Definition 4.1.1. Let $A \in \mathbf{Alg}_{\mathbb{E}_1}(\mathscr{C})$ and suppose that $\mathscr{C}$ is G-presentable, then we define the parametrized Thom object functor $\mathrm{Th}_G \colon \underline{\mathcal{S}}_{G/\underline{\mathrm{Pic}}_G(A)} \to \underline{\mathbf{LMod}}^G_A(\mathscr{C})$ as the parametrized Yoneda extension

$$\begin{tikzcd} \underline{\mathrm{Pic}}_G(A) \arrow[r] \arrow[d, hook, "\text{よ}_G"'] & \underline{\mathbf{LMod}}^G_A(\mathscr{C}) \\ \underline{\mathbf{PSh}}_G(\underline{\mathrm{Pic}}_G(A)) \arrow[ur, dashed, "\mathrm{Th}_G"'] & \end{tikzcd}$$

using $\underline{\mathcal{S}}_{G/\underline{\mathrm{Pic}}_G(A)} \simeq \underline{\mathbf{PSh}}_G(\underline{\mathrm{Pic}}_G(A))$ by parametrized straightening-unstraightening (Proposition 2.3.2).

Remark 4.1.2.

(i) Thom spectra are also often denoted by $\mathrm{Th}_G(f) = \mathrm{M}f$, motivated by classical examples such as $\mathrm{MO}, \mathrm{MU}, \mathrm{MSpin}, \cdots$ (and their equivariant versions). We will use this notation in Part II.

(ii) For $H \leq G$ we recover the non-parametrized H-Thom spectrum functor

$$\mathcal{S}_{H/\underline{\mathrm{Pic}}_H(A)} \to \mathbf{LMod}_{A_H}(\mathscr{C}_H),$$

which is given by the parametrized colimit

$$\mathrm{Th}_H(X \to \underline{\mathrm{Pic}}_H(A)) \simeq \underline{\mathrm{colim}}\left(X \to \underline{\mathrm{Pic}}_H(A) \to \underline{\mathbf{LMod}}^H_A(\mathscr{C})\right),$$

as demonstrated in [HHK+24, Corollary 5.2.3].

Example 4.1.3. The following also enhance multiplicatively.

(i) Let $c \colon X \to \underline{\mathrm{Pic}}_G(R)$ be equivalent to the trivial map picking out some $M \in \mathrm{Pic}_G(R)$. Its Thom object computed as the constant colimit, so $\mathrm{Th}_G(c) \simeq X \otimes M = \underline{\mathrm{colim}}_X M$.

(ii) One computes

$$\mathrm{Th}_G\left(X \times Y \to X \xrightarrow{f} \underline{\mathrm{Pic}}_G(A)\right) \simeq \mathrm{Th}_G(f) \otimes_A (A \otimes Y)$$

by symmetric monoidality of Th_G in the slice monoidal structure and (i). This does not need full parametrized (macrocosmic) monoidal straightening-unstraightening to pass from presheaves, instead we only need the classical statement [Ram22, Corollary 4.9] at the G-level.

We will now mimick the Thom spectrum definition (Definition 4.1.1) multiplicatively.

Construction 4.1.4. Let $(\mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes, \mathscr{C}^\otimes, A)$ be a distributive module datum (Definition 3.3.15). Then we define the $(\mathscr{O} \otimes \mathbb{E}_1)$-monoidal multiplicative G-Thom object functor

$$\mathrm{Th}_G^\otimes \colon \underline{\mathbf{PSh}}_G(\underline{\mathrm{Pic}}_G(A))^\otimes \to \underline{\mathbf{LMod}}_A^G(\mathscr{C})^\otimes$$

as the multiplicative parametrized Yoneda extension

$$\begin{tikzcd} \underline{\mathrm{Pic}}_G^\otimes(A) \arrow[r] \arrow[d, "よ_G^\otimes"'] & \underline{\mathbf{LMod}}_A^G(\mathscr{C})^\otimes \\ \underline{\mathbf{PSh}}_G(\underline{\mathrm{Pic}}_G(A))^\otimes \arrow[ur, dashed, "\mathrm{Th}_G^\otimes"'] & \end{tikzcd}$$

via an $(\mathscr{O} \otimes \mathbb{E}_1)$-monoidal version of [NS22, Corollary 6.0.12][9] using distributivity of $\mathbf{LMod}_A^G(\mathscr{C})^\otimes$ (see Theorem 3.3.14).

Since $\mathrm{Th}_G^\otimes$ is G-colimit preserving [NS22, Corollary 6.0.12], this extends the parametrized Thom spectrum construction (Definition 4.1.1).

We required an additional $\mathbb{E}_1$ to get Pic_G running (Construction 3.4.16), which is why we will always need $(\mathscr{O} \otimes \mathbb{E}_1)$ to get started. But once $\mathrm{Pic}_G^\otimes$ is constructed, we can restrict its multiplicative structure to potentially demand less structure from the other participating objects. That's why we will often carry around an G-∞-operad map $\mathscr{P}^\otimes \to \mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes$.

Corollary 4.1.5. Let $(\mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes, \mathscr{C}^\otimes, A)$ be a distributive module datum and consider a map $\mathscr{P}^\otimes \to \mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes$ in $\mathbf{Op}_{G,\infty}$. There is a functor

$$\mathrm{Th}_G^\otimes \colon \mathbf{Mon}_{\mathscr{P}}(\mathcal{S})_{/\,\mathrm{Pic}_G^\otimes(A)} \to \mathbf{Alg}_{\underline{\mathscr{P}/\mathscr{P}}}(\underline{\mathbf{LMod}}_A(\mathscr{C})) \to \mathbf{Alg}_{\underline{\mathscr{P}}}(\underline{\mathbf{LMod}}_A(\mathscr{C}))$$

extending the parametrized Thom spectrum functor Th_G.

Proof. We apply $\mathbf{Alg}_{\underline{\mathscr{P}/\mathscr{P}}}$ to $\mathrm{Th}_G^\otimes$ and use microcosmic straightening-unstraightening (Corollary 3.2.3) to identify the left side. The second functor is the forgetful functor. □

This shows the monoidality in maps $X^\otimes \to \underline{\mathrm{Pic}}_G^\otimes(A)$, which is not immediate from the Day convolution monoidal structure on $\underline{\mathbf{PSh}}_G(\underline{\mathrm{Pic}}_G(A))^\otimes$.

Remark 4.1.6. Our construction of the multiplicative parametrized Thom spectra is made to have good functoriality properties but there are other ways to obtain the multiplicative structure on parametrized Thom spectra. The following is one categorical level lower with the downside that it comes with less functoriality but with the upside of a different universal property, which is the one of particular interest to us in this article.

[9] The result is stated for G-symmetric monoidal G-∞-categories because they passed the adjunctions to G-commutative algebras. Passing to $\mathscr{O}$-algebras allows more general operads.

(i) Let $(\mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes, \mathscr{C}^\otimes, A)$ be a distributive module datum and let $\mathscr{P}^\otimes \to \mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes$ be a map in $\mathbf{Op}_{G,\infty}$. Given an $\mathscr{P}$-monoidal map $f\colon X^\otimes \to \underline{\mathrm{Pic}}_G^\otimes(A)$, we obtain the multiplicative Thom spectrum by operadic left Kan extension (Theorem 2.3.13)

$$\begin{tikzcd} X^\otimes \arrow[r, "f"] \arrow[d] & \underline{\mathrm{Pic}}_G^\otimes(A) \arrow[r] & \underline{\mathbf{LMod}}_A^G(\mathscr{C})^\otimes \\ \underline{\mathscr{P}}^\otimes, \arrow[urr, dashed, bend right, "\mathrm{Th}_G^\otimes(f)"'] & & \end{tikzcd}$$

which is the technique used in [ACB19] to obtain an $(\mathscr{O} \otimes \mathbb{E}_1)$-algebra structure on $\mathrm{Th}_G(f)$.

(ii) Mimicking [CCRY25, Proposition 7.8] with the language of parametrized higher category theory shows that the two monoidal structures on equivariant Thom spectra agree.

Having set up the multiplicative equivariant Thom spectrum functor, we can now give a universal property, which is an equivariant version of [ACB19, Theorem 3.5].

Remark 4.1.7. Let $\mathscr{O}^\otimes \in \mathbf{Op}_{G,\infty}$ and let $(\mathscr{C}^\otimes, R)$ be an $\mathscr{O}$-module datum. Then, consider an algebra $A \in \mathbf{Alg}_{\mathscr{O}}(\mathbf{LMod}_R)$. It comes with an $\mathscr{O}$-map from the initial object $R \to A$ by [NS22, Theorem 5.2.11(i)].

Theorem 4.1.8. Let $(\mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes, \mathscr{C}^\otimes, R)$ be a distributive module datum, $\mathscr{P}^\otimes \to \mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes$ be a map in $\mathbf{Op}_{G,\infty}$ and $f\colon X^\otimes \to \mathrm{Pic}_G^\otimes(R)$ be a map in $\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})$. Let $A \in \mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R)$. There is a natural equivalence

$$\mathrm{Map}_{\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R)}(\mathrm{Th}_G^\otimes(f), A) \simeq \mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})_{/\mathrm{Pic}_G^\otimes(R)}}\left(X, \mathrm{Pic}_G^\otimes(R)_{\downarrow A}\right),$$

i.e. there is an adjunction $\mathrm{Th}_G^\otimes(-) \dashv \mathrm{Pic}_G^\otimes(\mathscr{C})_{\downarrow(-)}$.

Proof. We write $i\colon \mathrm{Pic}_G^\otimes(R) \to \mathbf{LMod}_R^G(\mathscr{C})^\otimes$ and perform the following chain of natural equivalences:

$$\begin{aligned}
\mathrm{Map}_{\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R)}(\mathrm{Th}_G^\otimes(f), A) &\simeq \mathrm{Map}_{\mathbf{Fbrs}(\mathscr{P}^\otimes)_{/\mathbf{LMod}_R^G(\mathscr{C})^\otimes}}(X^\otimes, \mathbf{LMod}_R^G(\mathscr{C})^\otimes_{/A}) \\
&\simeq \mathrm{Map}_{\mathbf{Fbrs}(\mathscr{P}^\otimes)}(X^\otimes, \mathbf{LMod}_R^G(\mathscr{C})^\otimes_{/A}) \times_{\mathrm{Map}_{\mathbf{Fbrs}(\mathscr{P}^\otimes)}(X, \mathbf{LMod}_R^G(\mathscr{C})^\otimes)} \{if\} \\
&\simeq \mathrm{Map}_{\mathbf{Fbrs}(\mathscr{P}^\otimes)}(X^\otimes, \mathrm{Pic}_G^\otimes(R)_{\downarrow A}) \times_{\mathrm{Map}_{\mathbf{Fbrs}(\mathscr{P}^\otimes)}(X, \mathrm{Pic}_G^\otimes(R))} \{f\} \\
&\simeq \mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})}(X^\otimes, \mathrm{Pic}_G^\otimes(R)_{\downarrow A}) \times_{\mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})}(X, \mathrm{Pic}_G^\otimes(R))} \{f\} \\
&\simeq \mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})_{/\mathrm{Pic}_G^\otimes(R)}}(X^\otimes, \mathrm{Pic}_G^\otimes(R)_{\downarrow A}).
\end{aligned}$$

The first equivalence is Proposition 3.1.11. In the second and last equivalence we wrote out mapping spaces of slice ∞-categories. The third equivalence comes from pullback pasting

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathbf{Fbrs}(\mathscr{P}^\otimes)}(X^\otimes, \mathrm{Pic}_G^\otimes(R)_{\downarrow A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathbf{Fbrs}(\mathscr{P}^\otimes)}(X^\otimes, \mathbf{LMod}_R^G(\mathscr{C})^\otimes_{/A}) \arrow[d] \\ * \arrow[r, "f"'] & \mathrm{Map}_{\mathbf{Fbrs}(\mathscr{P}^\otimes)}(X^\otimes, \mathrm{Pic}_G^\otimes(R)) \arrow[r, "i_*"'] & \mathrm{Map}_{\mathbf{Fbrs}(\mathscr{P}^\otimes)}(X^\otimes, \mathbf{LMod}_R^G(\mathscr{C})^\otimes) \end{tikzcd}$$

where the right square is a pullback, hence the left one is a pullback if and only if the total rectangle is a pullback. The fourth equivalence is because the objects are space-valued fibrous patterns, so they are automatically coCartesian fibrations and maps automatically preserve coCartesian edges, which means that we may pass to $\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})$. $\square$

Colloquially, a map $\mathrm{Th}_G^{\otimes}(f) \to A$ corresponds to a lift

$$\begin{tikzcd} & \mathrm{Pic}_G^{\otimes}(R)_{\downarrow A} \arrow[d] \\ X \arrow[ur, dashed, "\exists!"] \arrow[r, "f"'] & \mathrm{Pic}_G^{\otimes}(R) \end{tikzcd}$$

in $\mathbf{Mon}_{\mathscr{O}}(\mathcal{S})$.

Construction 4.1.9. Let $\mathscr{O}^{\otimes}, \mathscr{Q}^{\otimes} \in \mathbf{Op}_{G,\infty}$ and $\mathscr{O}^{\otimes} \to \mathscr{Q}^{\otimes}$ be a map in $\mathbf{Op}_{G,\infty}$. Let $\underline{\mathscr{C}}^{\otimes}$ be a distributive $\mathscr{Q}$-monoidal G-∞-category. Then, $\mathrm{Res}_{\mathscr{O}}^{\mathscr{Q}}\colon \mathbf{Alg}_{\underline{\mathscr{Q}}}(\underline{\mathscr{C}}) \to \mathbf{Alg}_{\underline{\mathscr{O}}}(\underline{\mathscr{C}})$ preserves limits since limits are computed underlying [NS22, Theorem 5.1.3]. Moreover, these algebra categories are presentable by distributivity [NS22, Theorem 5.1.4(4)], so the adjoint functor theorem yields a left adjoint $\mathrm{Ind}_{\mathscr{O}}^{\mathscr{Q}}\colon \mathbf{Alg}_{\underline{\mathscr{O}}}(\underline{\mathscr{C}}) \to \mathbf{Alg}_{\underline{\mathscr{Q}}}(\underline{\mathscr{C}})$.

Corollary 4.1.10. Let $(\mathscr{O}^{\otimes} \otimes \mathbb{E}_1^{\otimes}, \mathscr{C}^{\otimes}, R)$ be a distributive module datum and consider a map $\mathscr{P}^{\otimes} \to \mathscr{O}^{\otimes} \otimes \mathbb{E}_1^{\otimes}$ in $\mathbf{Op}_{G,\infty}$. Consider a map $f : X^{\otimes} \to \mathrm{Pic}_G^{\otimes}(R)$ in $\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})$ and another map $\mathscr{P}^{\otimes} \to \mathscr{Q}^{\otimes} \to \mathscr{O}^{\otimes} \otimes \mathbb{E}_1^{\otimes}$ in $\mathbf{Op}_{G,\infty}$. Then, there is a natural equivalence

$$\mathrm{Ind}_{\mathscr{P}}^{\mathscr{Q}}(\mathrm{Th}_G^{\otimes}(f)) \simeq \mathrm{Th}_G^{\otimes}\left(\widetilde{f}\colon \mathrm{Ind}_{\mathscr{P}}^{\mathscr{Q}} X^{\otimes} \to \underline{\mathrm{Pic}}_G^{\otimes}(R)\right).$$

where $\widetilde{f}\colon \mathrm{Ind}_{\mathscr{P}}^{\mathscr{Q}} X^{\otimes} \to \underline{\mathrm{Pic}}_G^{\otimes}(R)$ is the $\mathscr{Q}$-map associated to $X^{\otimes} \to \underline{\mathrm{Pic}}_G^{\otimes}(R)$.

Proof. We wish to employ a Yoneda argument and proceed by using the universal property (Theorem 4.1.8). Indeed, we compute

$$\begin{aligned} \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_V}(\mathbf{LMod}_R)}\left(\mathrm{Ind}_{\mathbb{E}_0}^{\mathbb{E}_V} \mathrm{Th}_G^{\otimes}(f), A\right) &\simeq \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_0}(\mathbf{LMod}_R)}(\mathrm{Th}_G^{\otimes}(f), A) \\ &\simeq \mathrm{Map}_{\mathbf{Mon}_{\mathbb{E}_0}(\mathcal{S})_{/\mathrm{Pic}_G^{\otimes}(R)}}(X^{\otimes}, \mathrm{Pic}_G^{\otimes}(R)_{\downarrow A}) \\ &\simeq \mathrm{Map}_{\mathbf{Mon}_{\mathbb{E}_V}(\mathcal{S})_{/\mathrm{Pic}_G^{\otimes}(R)}}\left(\mathrm{Ind}_{\mathscr{P}}^{\mathscr{Q}} X^{\otimes}, \mathrm{Pic}_G^{\otimes}(R)_{\downarrow A}\right) \\ &\simeq \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_V}(\mathbf{LMod}_R)}(\mathrm{Th}_G^{\otimes}(\overline{f}), A). \end{aligned}$$

The third equivalence is obtained by writing out the mapping spaces of slice categories and then applying the universal property of $\mathrm{Ind}_{\mathscr{P}}^{\mathscr{Q}}$ by noting that on the right side we have grouplike objects. □

We will give a slight variant of this result in the grouplike setting in the next subsection (Corollary 4.2.7).

4.2 Abstract Orientation Theory

By definition (Definition 4.1.1), a Thom object is a G-colimit of some map $X \to \underline{\mathrm{Pic}}_G(R)$ of G-spaces, which can be viewed as a twisted version of $X \otimes R$ (see Example 4.1.3). Let $R \to A$ be another algebra map, along which we can base change

$$X \longrightarrow \underline{\mathrm{Pic}}_G(R) \xrightarrow{\mathrm{Ind}_R^A} \underline{\mathrm{Pic}}_G(A).$$

The base change could have had the effect of making this composite nullhomotopic, which ultimately leads to a detwisting of the Thom spectrum after base changing (Example 4.1.3). The detwisting is realized by such a nullhomotopy, which is then rightfully called *orientation*.

In this subsection, we will study the structure of multiplicative orientations along with the phenomena that it comes with – importantly, the multiplicative Thom isomorphism (Theorem 4.2.9). Throughout, we will always start with the same data, for which we introduce some additional terminology for the sake of brefity.

Notation 4.2.1. An orientation datum is a tuple $(\mathscr{O}^\otimes, \mathscr{P}^\otimes, \mathscr{C}, R, A, f)$ consisting of

- a distributive module datum $(\mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes, \mathscr{C}^\otimes, R)$; see Definition 3.3.15,
- a map $\mathscr{P}^\otimes \to \mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes$ in $\mathbf{Op}_{G,\infty}$,
- a map $f\colon X^\otimes \to \mathrm{Pic}_G^\otimes(R)$ in $\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})$,
- an algebra $A \in \mathbf{Alg}_{\mathscr{P}\otimes\mathbb{E}_1}(\mathscr{C})$ with a $\mathscr{P}^\otimes \otimes \mathbb{E}_1^\otimes$-map $R \to A$.[10]

In particular, $A \simeq R \otimes_R A \in \mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R)$.

Remark 4.2.2. We warn the reader that the necessary data for the results of this section can be quite confusing, see also [ACB19, Remark 3.7]. First, we start with an $(\mathscr{O} \otimes \mathbb{E}_2)$-ring R out of two reasons: Taking **LMod** kills one copy of $\mathbb{E}_1$ and to run our theory of Picard spaces (Corollary 3.4.12), we needed another copy of $\mathbb{E}_1$. Once we obtain these objects, we can freely restrict the operad structure which is why we consider a map $\mathscr{P}^\otimes \to \mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes$.

Next, we demand an algebra $A \in \mathbf{Alg}_{\mathscr{P}\otimes\mathbb{E}_1}(\mathscr{C})$ together with a $(\mathscr{P} \otimes \mathbb{E}_1)$-map $R \to A$. We needed to introduce an $\mathbb{E}_1$-again, so that the base change map Ind_R^A is a $\mathscr{P}$-map and this base change map is relevant for the theory of orientations. Note that this is different than the condition in Theorem 4.1.8 – we only needed a weaker statement there because the base change functor Ind_R^A did not show up.

We initiate the study of orientations through a universal example, parametrizing a program from Antolín-Camarena–Barthel [ACB19, Section 3.2].

Definition 4.2.3. Let $(\mathscr{O}^\otimes, \mathscr{P}^\otimes, \mathscr{C}, R, A, f)$ be an orientation datum. The space of $\mathscr{P}$-A-orientations of f is

$$\mathrm{Or}_A^{\mathscr{P}}(f) = \mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})_{/\mathrm{Pic}_G^\otimes(R)}}\left(f, \mathrm{GL}_1(\mathrm{Pic}^\otimes(R)_{\downarrow A}) \to \mathrm{Pic}_G^\otimes(R)\right).$$

We first show that this coincides with the approach from Antolín-Camarena–Barthel involving their $B(R, A)$ [ACB19, Definition 3.12].

Lemma 4.2.4. Let $(\mathscr{O}^\otimes, \mathscr{P}^\otimes, \mathscr{C}, R, A, f)$ be an orientation datum and $H \leq G$, then the underlying space of $\mathrm{GL}_1(\mathrm{Pic}^\otimes(R)_{\downarrow A})^H$ is the subgroupoid of $\mathrm{Pic}(R_H)_{\downarrow A_H}$ consisting of R_H-module maps $M \to A_H$ such that the adjoint $\mathrm{Ind}_{R_H}^{A_H} M \to A_H$ is an equivalence.

The latter is also called $B(R_H, A_H)$ in the language of Antolín-Camarena–Barthel [ACB19, Definition 3.12].

Proof. Since this is a completely levelwise statement, it amounts to an entirely classical check $\mathrm{GL}_1(\mathrm{Pic}^\otimes(R)_{\downarrow A}) \simeq B(R, A)$ for the subgroup $H = e$. To do so, we use the equivalence $\mathrm{GL}_1(\mathrm{Pic}^\otimes(R)_{\downarrow A}) \simeq \mathrm{Pic}(R) \times_{\mathrm{Pic}\, A} *$ from Corollary 3.4.19, which is $B(R, A)$, as noted in the proof of [ACB19, Proposition 3.16]. □

The upshot is that the GL_1-business allows us to bypass some technical checks in parametrized higher category theory such as $B(R, A)$ admitting a parametrized version which furthermore admits a parametrized multiplicative enhancement. All of this was already done by our hard work involving GL_1.

Let us now demonstrate that this recovers the classical interpretations of (multiplicative) orientations.

[10] Here, R was by definition $\mathscr{O}^\otimes \otimes \mathbb{E}_2^\otimes$ which we restrict to $\mathscr{P}^\otimes \otimes \mathbb{E}_1^\otimes$.

Corollary 4.2.5. Let $(\mathscr{O}^{\otimes}, \mathscr{P}^{\otimes}, \mathscr{C}, R, A, f)$ be an orientation datum. The following are equivalent characterizations of $\mathscr{P}$-A-orientations:

(i) A $\mathscr{P}$-lift

$$\begin{tikzcd} & \mathrm{GL}_1(\mathrm{Pic}_G^{\otimes}(R)_{\downarrow A}) \arrow[d] \\ X^{\otimes} \arrow[ur, dashed] \arrow[r, "f"'] & \mathrm{Pic}_G^{\otimes}(R) \end{tikzcd}$$

of f.

(ii) A nullhomotopy of the composite

$$X^{\otimes} \xrightarrow{f} \mathrm{Pic}_G^{\otimes}(R) \xrightarrow{\mathrm{Ind}_R^A} \mathrm{Pic}_G^{\otimes}(A).$$

in $\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})$.

(iii) A map $\mathrm{Th}_G^{\otimes}(f) \to A$ in $\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R)$ such that for every $x\colon * \to X$ the adjoint A-module map corresponding to the R-module map

$$\mathrm{Th}_G(f \circ x) \longrightarrow \mathrm{Th}_G(f) \longrightarrow A$$

is an equivalence.

Proof.

(i) This is Definition 4.2.3 (after unravelling the mapping spaces of slice ∞-categories).

(ii) This follows from $\mathrm{GL}_1(\mathrm{Pic}_G^{\otimes}(R)_{\downarrow A}) \simeq \mathrm{Pic}_G^{\otimes}(R) \times_{\mathrm{Pic}_G^{\otimes}(A)} *$ (see Corollary 3.4.19).

(iii) This follows from our explicit description of $\mathrm{GL}_1(\mathrm{Pic}_G^{\otimes}(R)_{\downarrow A})$ (see Lemma 4.2.4) and the universal property of multiplicative parametrized Thom spectra (Theorem 4.1.8).

□

The description (ii) already featured in the intro of this subsection and (iii) lifts the classical notion of Thom classes.

Lemma 4.2.6. Let $(\mathscr{O}^{\otimes}, \mathscr{P}^{\otimes}, \mathscr{C}, R, A, f)$ be an orientation datum. Let $X \in \mathbf{Mon}_{\mathscr{P}}(\mathcal{S})$. Suppose one of the following conditions.

(i) Assume that X is levelwise connected.

(ii) Assume that there is a map $\mathbb{E}_1^{\otimes} \to \mathscr{O}^{\otimes}$ and that X is grouplike.

Then, $\mathrm{Or}_A^{\mathscr{P}}(f) \simeq \mathrm{Map}_{\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R(\mathscr{C}))}(\mathrm{Th}_G^{\otimes}(f), A)$.

Proof. By the universal property (Theorem 4.1.8) of $\mathrm{Th}_G^{\otimes}$ we have

$$\mathrm{Map}_{\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R(\mathscr{C}))}(\mathrm{Th}_G^{\otimes}(f), A) \simeq \mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})_{/\mathrm{Pic}_G^{\otimes}(R)}}\left(X, \mathrm{Pic}_G^{\otimes}(R)_{\downarrow A}\right)$$

so we need to show that every map $X \to \mathrm{Pic}_G^{\otimes}(R)_{\downarrow A}$ factors through $\mathrm{GL}_1(\mathrm{Pic}_G^{\otimes}(R)_{\downarrow A})$ for grouplike X. This is a levelwise statement which with Lemma 4.2.4 becomes [ACB19, Lemma 3.15]. □

With this we can give a variation of Corollary 4.1.10.

Corollary 4.2.7. Let $(\mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes, \mathscr{C}^\otimes, R)$ be a distributive module datum. Let X be a (levelwise) connected pointed G-space and consider a map $f\colon X \to \underline{\mathrm{Pic}}_G(R)$ of pointed G-spaces. Suppose that there is a map $\mathbb{E}_V^\otimes \to \mathscr{O}^\otimes \otimes \mathbb{E}_1^\otimes$. Then, there is a natural equivalence

$$\mathrm{Ind}_{\mathbb{E}_0}^{\mathbb{E}_V}\left(\mathrm{Th}_G^\otimes(f)\right) \simeq \mathrm{Th}_G^\otimes\left(\overline{f}\colon \Omega^V\Sigma^V X \to \underline{\mathrm{Pic}}_G(R)\right)$$

where $\overline{f}$ is the V-fold loop map extension [Jur25, Theorem 3.15].

Proof. The proof is essentially the same as in Corollary 4.1.10 besides the new ingredients coming from grouplikeness. We include the full proof for the convenience of the reader.

The equivariant approximation theorem [Jur25, Theorem 3.15] allows us to define $\overline{f}$ as stated. Let $A \in \mathbf{Alg}_{\mathbb{E}_V}(\mathbf{LMod}_R)$. We wish to employ a Yoneda argument and proceed by using the universal property (Theorem 4.1.8). Indeed, we compute

$$\begin{aligned}
\mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_V}(\mathbf{LMod}_R)}\left(\mathrm{Ind}_{\mathbb{E}_0}^{\mathbb{E}_V}\mathrm{Th}_G^\otimes(f), A\right) &\simeq \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_0}(\mathbf{LMod}_R)}(\mathrm{Th}_G^\otimes(f), A)\\
&\simeq \mathrm{Map}_{\mathbf{Mon}_{\mathbb{E}_0}(\mathcal{S})_{/\mathrm{Pic}_G^\otimes(R)}}(X, \mathrm{GL}_1(\mathrm{Pic}_G^\otimes(R)_{\downarrow A}))\\
&\simeq \mathrm{Map}_{\mathbf{Mon}_{\mathbb{E}_V}(\mathcal{S})_{/\mathrm{Pic}_G^\otimes(R)}}\left(\Omega^V\Sigma^V X, \mathrm{GL}_1(\mathrm{Pic}_G^\otimes(R)_{\downarrow A})\right)\\
&\simeq \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_V}(\mathbf{LMod}_R)}(\mathrm{Th}_G^\otimes(\overline{f}), A).
\end{aligned}$$

The third equivalence is obtained by writing out the mapping spaces of slice categories and then applying the universal property of $\Omega^V\Sigma^V$ by noting that on the right side we have grouplike objects. The second and fourth equivalence use the grouplike/connectedness condition (Lemma 4.2.6). $\square$

Furthermore, Lemma 4.2.6 gives rise to a simple example, which is surprisingly the most important orientation for this article.

Example 4.2.8. Let $(\mathscr{O}^\otimes, \mathscr{P}^\otimes, \mathscr{C}, R, A, f)$ be an orientation datum. Suppose that $X^\otimes$ is grouplike. Then, $\mathrm{id}_{\mathrm{Th}_G^\otimes(f)}\colon \mathrm{Th}_G^\otimes(f) \to \mathrm{Th}_G^\otimes(f)$ is a $\mathscr{P}$-$\mathrm{Th}_G^\otimes(f)$-orientation.

Theorem 4.2.9. Let $(\mathscr{O}^\otimes, \mathscr{P}^\otimes, \mathscr{C}, R, A, f)$ be an orientation datum. A $\mathscr{P}$-A-orientation of f gives rise to a Thom isomorphism

$$A \otimes_R \mathrm{Th}_G^\otimes(f) \simeq A \otimes X^\otimes$$

in $\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_A)$.

Proof. Since left Kan extensions commute with left adjoints, we conclude

$$A \otimes_R \mathrm{Th}_G^\otimes(f) \simeq \mathrm{Th}_G^\otimes(A \otimes_R f)$$

in $\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_A)$. On the other hand, an orientation is a nullhomotopy of $A \otimes_R f$ (see Corollary 4.2.5 (ii)). We can write this null map as $\mathrm{Ind}_{\mathbb{1}}^R \circ c$ where $c\colon X^\otimes \to \mathrm{Pic}_G(\mathbb{1})$ is the null map. Thus, it is equivalent to

$$A \otimes_R \mathrm{Th}_G^\otimes(f) \simeq \mathrm{Th}_G^\otimes(A \otimes_R f) \simeq \mathrm{Ind}_{\mathbb{1}}^A(\mathbb{1} \otimes X^\otimes) \simeq A \otimes X^\otimes.$$

by Example 4.1.3. $\square$

Corollary 4.2.10. Let $(\mathscr{O}^\otimes, \mathscr{P}^\otimes, \mathscr{C}, R, A, f)$ be an orientation datum.

(i) There is a pullback square

$$\begin{tikzcd}
\mathrm{Or}^{\mathscr{P}}_A(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "\mathrm{const}_A"] \\
* \arrow[r, "(R\otimes_A -)_* \circ f"'] & \mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})}(X^{\otimes}, \mathrm{Pic}^{\otimes}_G(A))
\end{tikzcd}$$

In particular, it is either empty or $\Omega \,\mathrm{Map}_{\mathscr{P}}(X^{\otimes}, \mathrm{Pic}^{\otimes}_G(A))$.

(ii) Suppose that $X^{\otimes}$ is furthermore grouplike. Then, $\mathrm{Map}_{\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R)}(\mathrm{Th}^{\otimes}_G(f), A)$ is empty or

$$\mathrm{Map}_{\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R)}(\mathrm{Th}^{\otimes}_G(f), A) \simeq \mathrm{Map}_{\mathbf{Alg}_{\mathscr{P}}(\mathbf{LMod}_R)}(R \otimes X, A).$$

This equivalence is also true for parametrized mapping spaces.

Proof.

(i) This follows immediately from pullback pasting

$$\begin{tikzcd}
\mathrm{Or}^{\mathscr{P}}_A(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})}(X^{\otimes}, \mathrm{GL}_1(\mathrm{Pic}^{\otimes}_G(R)_{\downarrow A})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "\mathrm{const}_A"] \\
* \arrow[r, "f"'] & \mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})}(X^{\otimes}, \mathrm{Pic}^{\otimes}_G(R)) \arrow[r, "(R\otimes_A -)_*"'] & \mathrm{Map}_{\mathbf{Mon}_{\mathscr{P}}(\mathcal{S})}(X^{\otimes}, \mathrm{Pic}^{\otimes}_G(A))
\end{tikzcd}$$

The left square is a pullback by definition of $\mathrm{Or}^{\mathscr{P}}_A(-)$ and the universal property of slice ∞-categories. The right square is a pullback by Corollary 3.4.19. Thus, the composite rectangle is a pullback.

(ii) By Lemma 4.2.6 we are analyzing the space of orientations. If there is one, then the monoidal Thom isomorphism (Theorem 4.2.9) yields $A \otimes_R \mathrm{Th}^{\otimes}_G(f) \simeq A \otimes X$, so an adjunction argument gives the desired mapping space equivalence.

□

Part II
Applications

Section 5 This section sets up the main organizational and computational prerequisites for our obstruction theory.

In Section 5.1 we review Hill–Meier's notion of strongly even spectra and sharpen some of the basic results.

In Section 5.2 we extend this to the setting of towers of G-spectra. The expert is free to skip this section by agreeing that, under suitable conditions, the limit of a tower of strongly even spectra is strongly even.

In Section 5.3 we study a cohomological version of the equivariant slice tower. We start with a quick review of the slice tower; then we set up the cohomological slice tower and finish with providing criteria to check when the cohomological slice tower is strongly even.

Section 6 This section sets up our main obstruction theory for studying structured orientations of equivariant Thom spectra. The main idea in this section is that: the multiplicative Thom isomorphism, combined with the recognition principle, reduces the study of multiplicative orientations to Bredon computations of deloopings.

In Section 6.1 we produce an obstruction theory for lifting non-equivariant $\mathbb{E}_n$-orienations to equivariant $\mathbb{E}_V$-orientations.

In Section 6.2 we specialize to studying self-orientations of equivariant Thom spectra. We give a criterion for lifting multiplicative splittings of Thom spectra to multiplicative splittings of equivariant Thom spectra.

Section 7 This section proves our main results. We apply the the obstruction theory developed in the previous sections ot the study of multiplicative Real orientations.

In Section 7.1 we produce $\mathbb{E}_\rho$-orientations for strongly even $\mathbb{E}_\infty^{C_2}$-rings. We do this by producing an example in a versal case through Real Wilson spaces.

In Section 7.2, armed with the existence of orientations from the previous section, we deploy our obstruction theory to prove Theorem E, namely that underlying complex orientations $\mathrm{MU} \to E^e$ of strongly even $\mathbb{E}_\infty^{C_2}$-ring spectra lift to $\mathbb{E}_\rho$-Real orientations $\mathrm{MU}_\mathbb{R} \to E$. We use this to enhance various orientations of interest like the Hahn–Shi Real orientations of Lubin–Tate theory. For $C_2 \leq G$ we discuss an $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-structure on the normed versions.

In Section 7.3, we produce an $\mathbb{E}_\rho$-multiplication on $\mathrm{BP}_\mathbb{R}$, proving Theorem H from the introduction. We also produce $\mathbb{E}_\rho$-Adams operations on $\mathrm{BP}_\mathbb{R}$.

Section 8 This section considers applications of equivariant Thom spectra outside the realm of structured orientations.

In Section 8.1 we prove formulas for the equivariant factorization homology of equivariant Thom spectra. We work in the generality of R-module Thom spectra. In particular, this specializes to give formulas for relative Real Topological Hochschild Homology $\mathrm{THR}(-/R)$.

In Section 8.2 we study $\mathbb{E}_V$-quotients via equivariant Thom spectra. In particular we rephrase Levy's equivariant Hopkins–Mahowald Theorem in terms of $\mathbb{E}_V$-quotients.

In Section 8.3 we prove some basic facts about nilpotence for $\mathbb{E}_V$-algebras. We show that $\mathrm{MU}_\mathbb{R}$ detects nilpotence for $\mathbb{E}_\sigma$-rings. We show that $\mathrm{H}\underline{\mathbb{Z}}$ detects nilpotence for $\mathbb{E}_\sigma \otimes \mathbb{E}_\infty$-rings

As a guide to Part II we include a sketch proof of Theorem H. Although Theorem H follows formally from Theorem E, exhibiting the proof strategy directly for Theorem H helps to quarantine certain technical aspects from the key ideas. We hope this also acts as a reasonable summary of the parameterized theory from Part I for the reader who skipped directly to Part II.

Theorem G (Theorem 7.3.1)**.** The C_2-spectrum $\mathrm{BP}_{\mathbb{R}}$ admits an $\mathbb{E}_\rho$-algebra structure. Let $G \geq C_2$. The G-spectrum $\mathrm{BP}^{((G))} := N_{C_2}^G \mathrm{BP}_{\mathbb{R}}$ admits a $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-algebra structure

Proof Sketch. The $\mathrm{Coind}_{C_2}^G \mathbb{E}_\rho$-algebra structure on $\mathrm{BP}^{((G))}$ follows formally once we produce an $\mathbb{E}_\rho$-algebra structure on $\mathrm{BP}_{\mathbb{R}}$ (Construction 2.2.5). So we focus on $\mathrm{BP}_{\mathbb{R}}$.

The Real Brown–Peterson spectrum $\mathrm{BP}_{\mathbb{R}}$ can be defined as a sequential colimit along iterates of a (Real) equivariant version of Quillen's idempotent [Ara79, Section 7]. Since filtered colimits of algebras are computed on underlying, to produce an $\mathbb{E}_\rho$-algebra structure on $\mathrm{BP}_{\mathbb{R}}$, it suffices to lift Quillen's idempotent to an $\mathbb{E}_\rho$-algebra map $\mathrm{MU}_{\mathbb{R}} \to \mathrm{MU}_{\mathbb{R}}$.

For the reader unfamiliar with parameterized higher category theory, for the purpose of this sketch, the most important aspect – aside from the parameterized theory being essential to constructing a multiplicative equivariant Thom isomorphism (Theorem 4.2.9) – is that mapping spaces are replaced with mapping C_2-spaces. In particular, the space of $\mathbb{E}_\rho$-algebra maps $\mathrm{MU}_{\mathbb{R}} \to \mathrm{MU}_{\mathbb{R}}$ refines to a C_2-space

$$\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}});$$

the C_2-fixed points is precisely the space of $\mathbb{E}_\rho$-algebra maps $\mathrm{MU}_{\mathbb{R}} \to \mathrm{MU}_{\mathbb{R}}$, i.e.

$$\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}})^{C_2} \simeq \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}});$$

the underlying non-equivariant space is precisely the space of $\mathbb{E}_2$-algebra maps $\mathrm{MU} \to \mathrm{MU}$

$$\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}})^{e} \simeq \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_2}(\mathbf{Sp})}(\mathrm{MU}, \mathrm{MU});$$

and the map induced be the inclusion of the fixed points

$$\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}})^{C_2} \to \underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}})^{e}$$

is precisely the restriction map

$$\mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}}) \to \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_2}(\mathbf{Sp})}(\mathrm{MU}, \mathrm{MU}).$$

This parameterized perspective motivates a natural strategy: lift Chadwick–Mandell's $\mathbb{E}_2$-version of Quillen's idempotent $\mathrm{MU} \to \mathrm{MU}$ [CM15, Theorem 1.2] along the above restriction to an $\mathbb{E}_\rho$-map $\mathrm{MU}_{\mathbb{R}} \to \mathrm{MU}_{\mathbb{R}}$. In particular, to produce a such a lift, it suffices to show that the restriction map in the Mackey functor $\underline{\pi}_0\, \underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}})$ is a surjective; we will in fact show this is an isomorphism.

Using the multiplicative parameterized Thom isomorphism (Theorem 4.2.9), we make this computation more tractable. We have the following equivalences of C_2-spaces:

$$\begin{aligned}\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}}) &\simeq \underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\Sigma^\infty_+ \mathrm{BU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}}) \\ &\simeq \underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_\rho}(\underline{\mathcal{S}}_{C_2})}(\mathrm{BU}_{\mathbb{R}}, \Omega^\infty \mathrm{MU}_{\mathbb{R}}) \\ &\simeq \underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}^{\mathrm{grp}}_{\mathbb{E}_\rho}(\underline{\mathcal{S}}_{C_2})}(\mathrm{BU}_{\mathbb{R}}, \mathrm{GL}_1\, \mathrm{MU}_{\mathbb{R}})\end{aligned}$$

The last equivalence uses that $\mathrm{BU}_{\mathbb{R}}$ is grouplike, and the adjunction involving GL_1 (Proposition 3.4.8). The reason this reduction is useful is we can now use equivariant loop space theory and the equivariant recognition principle [GM17b, CHLL24, Jur25]. We have further equivalences of C_2-spaces

$$\begin{aligned}\underline{\mathrm{Map}}_{\mathbf{Alg}^{\mathrm{grp}}_{\mathbb{E}_\rho}(\underline{\mathcal{S}}_{C_2})}(\mathrm{BU}_{\mathbb{R}}, \mathrm{GL}_1\,\mathrm{MU}_{\mathbb{R}}) &\simeq \underline{\mathrm{Map}}_{\underline{\mathcal{S}}_*^{C_2}}(\mathrm{B}^\rho\mathrm{BU}_{\mathbb{R}}, \mathrm{B}^\rho\,\mathrm{GL}_1\,\mathrm{MU}_{\mathbb{R}}) \\ &\simeq \underline{\mathrm{Map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{B}^\rho\mathrm{BU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}}) \\ &\simeq \underline{\mathrm{Map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}})\end{aligned}$$

The first equivalence is the equivariant recognition principle (Theorem 3.4.3). For the second, we use the equivalence between equivariant infinite loop spaces and connective equivariant spectra (Theorem 3.4.3). The third equivalence uses equivariant Bott periodicity (Example 3.4.5).

In summary, we have reduced the problem to computing the $\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}}$-cohomology of $\mathrm{BSU}_{\mathbb{R}}$. More precisely, to produce an $\mathbb{E}_\rho$-lift of Quillen's idempotent, it suffices to show that the restriction map

$$\mathrm{res}^{C_2}_e\colon \pi_0^{C_2}(\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}})) \to \pi_0^{e}(\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}}))$$

is surjective. However, it is no harder, and in fact it is more suggestive of a strategy, to show that the restriction maps

$$\mathrm{res}^{C_2}_e\colon \pi_{*\rho}^{C_2}(\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}})) \to \pi_{2*}^{e}(\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}}))$$

are isomorphisms; it suggests to approach this by showing $\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1(\mathrm{MU}_{\mathbb{R}}))$ is a strongly even C_2-spectrum in the sense of Hill–Meier (Definition 5.1.3).

Our goal is to reduce the claim that $\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1(\mathrm{MU}_{\mathbb{R}}))$ is strongly even to a combination known computations, namely: the fact that $\mathrm{MU}_{\mathbb{R}}$ is strongly even; and that the Bredon cohomology of $\mathrm{BSU}_{\mathbb{R}}$ is strongly even. The two ideas we use to achieve this reduction are: setting up a cohomological version of the slice tower (Section 5.3); and extending the Hill–Meier's notion of strongly even C_2-spectra to the setting of filtered, or equivalently towers of C_2-spectra (Definition 5.2.1).

The cohomological slice tower is a cohomogical version of the homological slice tower studied in Carrick–Hill–Ravenel [CHR24]. For $\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}})$, the corresponding cohomological slice tower (Construction 5.3.4) is

$$\mathrm{CST}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}; \Sigma^\rho\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}}) = \underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, P^{\leq\bullet}\Sigma^\rho\mathrm{gl}_1\mathrm{MU}_{\mathbb{R}}).$$

By construction, the associated graded of this tower compute Bredon cohomology with coefficients in the slices of $\mathrm{MU}_{\mathbb{R}}$. By extending Hill–Meier's notion of strongly evenness to towers of C_2-spectra (Definition 5.2.1), we show that, under certain evenness conditions, the limit of a tower of C_2-spectra is strongly even if its associated graded pieces are strongly even (Proposition 5.2.4). Hence, applied to the cohomological slice tower, we indeed achieve the desired reduction and learn that $\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^{C_2}}(\Sigma^\infty\mathrm{BSU}_{\mathbb{R}}, \Sigma^\rho\mathrm{gl}_1(\mathrm{MU}_{\mathbb{R}}))$ is strongly even.

Thus, we have sketched that the restriction

$$\pi_{*\rho}^{C_2}\underline{\mathrm{Map}}_{\mathbf{Alg}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}}) \to \pi_{2*}^{e}\underline{\mathrm{Map}}_{\mathbf{Alg}_{\mathbb{E}_\rho}(\underline{\mathbf{Sp}}_{C_2})}(\mathrm{MU}_{\mathbb{R}}, \mathrm{MU}_{\mathbb{R}})$$

is an isomorphism, whence producing an $\mathbb{E}_\rho$-lift of Quillen's idempotent. □

5 Strongly Even Towers, & the Cohomological Slice Tower

The essential idea in this section is the following: In Real oriented homotopy theory, Hill–Meier's notion of strong eveness allows many (Real) equivariant statements to be deduced from underlying non-equivariant statements; by extending Hill–Meier's notion to the setting of towers of spectra, we can reduce equivariant obstruction theory to ordinary obstruction theory.

Although elementary, we expect the results here to be of independent interest, so we develop more than is strictly needed to prove Theorem H and Theorem E.

In the case one is only interested in Theorem H and Theorem E, the expert is free to skip this section by agreeing that the restriction map

$$\mathrm{res}^{C_2}_e\colon \pi^{C_2}_{*\rho}(\underline{\mathrm{map}}_{\mathbf{Sp}^{C_2}}(X,E)) \to \pi_{2*}(\mathrm{map}_{\mathbf{Sp}^{C_2}}(X,E))$$

is an equivalence when: E is strongly even, and slice bounded below; and $\mathrm{H}\underline{\mathbb{Z}} \otimes X$ is of finite type, is $\mathrm{H}\underline{\mathbb{Z}}$-free, and has a basis given by spheres in dimensions $n\rho$ (see Proposition 5.3.8).

5.1 Strongly Even Spectra

Evenness plays a key role in chromatic homotopy theory. Since $\mathbb{CP}^\infty$ is built from even cells S^{2n}, there is no obstruction to building complex orientations for even ring spectra. Hill–Meier [HM17] introduced the following notion of equivariant evenness suited for Real oriented homotopy theory.

Definition 5.1.1 (Hill–Meier)**.** Let $X \in \mathbf{Sp}_{C_2}$. It is said to be Real even[11] if $\pi^{C_2}_{n\rho-1}(X)$ and $\pi^e_{2n-1}(X)$ are both zero for all n.

Let $\mathbb{CP}^\infty_{\mathbb{R}}$ denote $\mathbb{CP}^\infty$ equipped with the C_2-action by complex conjugation. In the same way that complex orientations are determined by $\mathbb{CP}^\infty$, Real orientations are determined by $\mathbb{CP}^\infty_{\mathbb{R}}$. Since $\mathbb{CP}^\infty_{\mathbb{R}}$ is built out of cells of the form $S^{n\rho}$, the following is can be deduced from obstruction theory analogous to the classical statement.

Lemma 5.1.2 ([HM17, Lemma 3.3])**.** Let X be a Real even C_2-spectrum. Then X admits a Real orientation.

Hill–Meier also introduced a stronger notion of Real evenness, called strong evenness. The main motivation for this definition is that, in examples of interest, often the $*\rho$-graded homotopy groups contain more relevant information than the integer graded homotopy groups. For example: for $\mathrm{KU}_{\mathbb{R}}$, the equivariant Bott class lives in degree ρ; for $\mathrm{MU}_{\mathbb{R}}$, the Lazard ring lives in degrees $*\rho$.

Definition 5.1.3 (Hill–Meier)**.** Let $X \in \mathbf{Sp}_{C_2}$. We say that X is strongly even[12] if the following conditions are met:

(i) We have both $\pi^e_{2*-1}(X) = 0$ and $\pi^{C_2}_{*\rho-1}(X) = 0$, i.e. X is Real even.

(ii) The restriction maps $\mathrm{res}^{C_2}_e : \pi^{C_2}_{*\rho}(X) \to \pi^e_{2*}(X)$ are isomorphisms.

Example 5.1.4. All of the C_2-spectra $\mathrm{MU}_{\mathbb{R}}$, $\mathrm{ku}_{\mathbb{R}}$, $\mathrm{tmf}_1(n)$, $\mathrm{BP}_{\mathbb{R}}$, $\mathrm{BP}_{\mathbb{R}}\langle n\rangle$, and the Lubin–Tate theories E_n are strongly even, see [HK01, HM17, GM17a, HS20].

One of the main benefits of working with strongly even spectra is that many equivariant statements can then be deduced from the non-equivariant counterparts. For example, the following is particularly useful.

[11] Hill–Meier simply use the term even; we add the prefix Real to avoid overloading the term even.
[12] We were strongly tempted to instead use the term *Really even*.

Lemma 5.1.5 ([HM17, Lemma 3.4])**.** If $f\colon X \to Y$ is a map between strongly even spectra such that $f^e\colon X^e \to Y^e$ is an equivalence, then f is an equivalence.

A notable feature of strongly even spectra is that they all poses a 'gap' in their $RO(C_2)$-graded homotopy groups. Greenlees showed this in fact provides an equivalent reformulation of strong eveness [Gre18].

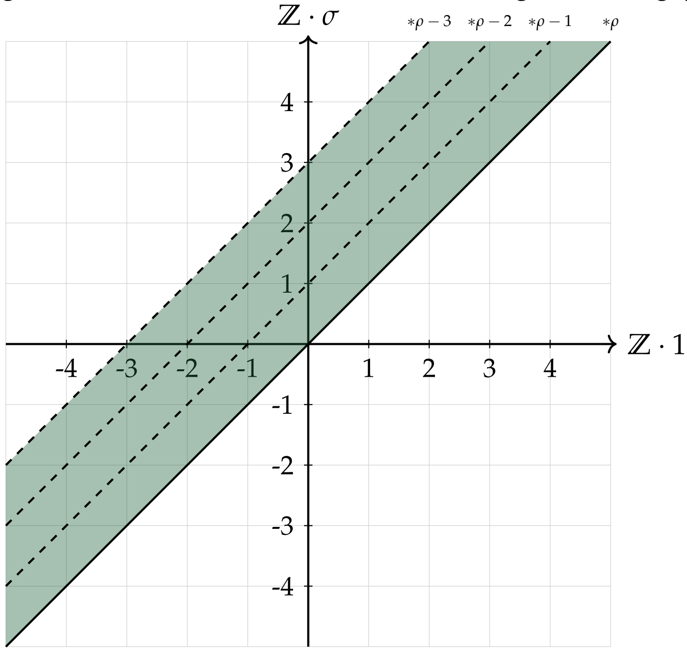
Lemma 5.1.6 (Greenlees' gap characterization)**.** Let $X \in \mathbf{Sp}_{C_2}$. The following are equivalent.

(i) X is strongly even.

(ii) $\pi^{C_2}_{*\rho-i}(X) = 0$ for $i = 1, 2, 3$.

Proof. See the proof of [Gre18, Lemma 1.2]. To read his statement note that Greenlees' definition of strongly even does not include the condition that the underlying non-equivariant spectrum is even. □

A caricature of the gap, heavily inspired by the diagrams in [Gre18], is illustrated below. The homotopy groups along the dotted lines vanish. The shaded region is the gap.



The gap characterization is a convenient way to check if a C_2-spectrum is strongly even. The following example will be used repeatedly in our applications.

Example 5.1.7. Let R be a strongly even $\mathbb{E}_\infty^{C_2}$-ring spectrum. Then $\mathrm{gl}_1(R)$ is a strongly even C_2-spectrum.

Proof. Using Greenlees' gap characterization (Lemma 5.1.6), this follows almost immediately from the computation of $\pi_V^{C_2}(\mathrm{GL}_1 R)$, see Lemma 3.4.14. The only potential subtlety is the case $V = \rho - 1 = \sigma$ since $\dim(\sigma^{C_2}) = 0$.

To handle this, we consider the fiber sequence $C_2/e_+ \to S^0 \to S^\sigma$ which gives rise to a long exact sequence

$$\cdots \longrightarrow \pi_1^{C_2}(X) \longrightarrow \pi_1^e(X) \longrightarrow \pi_\sigma^{C_2}(X) \longrightarrow \pi_0^{C_2}(X) \longrightarrow \pi_0^e(X) \longrightarrow \cdots$$

for every C_2-spectrum X. This gives a short exact sequence

$$0 \longrightarrow \operatorname{coker}\left(\pi_1^{C_2}(X) \to \pi_1^e(X)\right) \longrightarrow \pi_\sigma^{C_2}(X) \longrightarrow \ker\left(\pi_0^{C_2}(X) \to \pi_0^e(X)\right) \longrightarrow 0.$$

For $X = \mathrm{gl}_1(R)$ this becomes

$$0 \longrightarrow \mathrm{coker}\left(\pi_1^{C_2}(R) \to \pi_1^e(R)\right) \longrightarrow \pi_\sigma^{C_2}(\mathrm{gl}_1(R)) \longrightarrow \ker\left(\pi_0^{C_2}(R)^\times \to \pi_0^e(R)^\times\right) \longrightarrow 0.$$

Since R is strongly even, the restriction $\pi_0^{C_2}(R) \to \pi_0^e(R)$ is an isomorphism, and $\pi_1^e(R) = 0$. Hence, $\pi_\sigma^{C_2}(\mathrm{gl}_1(R)) = 0$. □

The following observation is elementary but will be used repeatedly. It's a more precise gap behaviour than typically noted – namely, provided that the underlying non-equivariant spectrum is even, the vanishing of just $\pi_{*\rho-2}^{C_2}(X)$ is equivalent to strong evenness.

Lemma 5.1.8. Let $X \in \mathbf{Sp}_{C_2}$. Suppose that following conditions hold:

(i) For all n we have $\pi_{2n-1}^e(X) = 0$.

(ii) For all n we have $\pi_{n\rho-2}^{C_2}(X) = 0$.

Then, the following conditions also hold:

(iii) For all n we have $\pi_{n\rho-1}^{C_2}(X) = 0$.

(iv) For all n we have $\pi_{n\rho-3}^{C_2}(X) = 0$.

In particular, X is strongly even.

Proof. First we show that (i) and (ii) imply (iii), then we show that (i) and (iii) imply (iv). Both of these will follow from considerations of the cofiber sequence

$$C_2/e_+ \longrightarrow S^0 \longrightarrow S^\sigma$$

and a part of the resulting long exact sequence

$$\pi_{a+b+1}^e(X) \longrightarrow \pi_{a+(b+1)\sigma}^{C_2}(X) \longrightarrow \pi_{a+b\sigma}^{C_2}(X) \xrightarrow{\mathrm{res}_e^{C_2}} \pi_{a+b}^e(X)$$

for $a, b \in \mathbb{Z}$. Indeed:

(iii) Take $a = n-1$ and $b = n$. Then, the latter three terms are given by

$$\pi_{n\rho-2}^{C_2}(X) \longrightarrow \pi_{n\rho-1}^{C_2}(X) \longrightarrow \pi_{2n-1}^e(X)$$

and the outer two are 0 by (1) and (2), so the middle term is also 0.

(iv) Take $a = n-1$ and $b = n+1$. Then, the first three terms are given by

$$\pi_{2n+1}^e(X) \longrightarrow \pi_{(n+2)\rho-3}^{C_2}(X) \longrightarrow \pi_{n\rho-2}^{C_2}(X)$$

and the outer two are 0 by (i) and (ii), so the middle term is also 0.

Greenlees' gap characterization of strong evenness Lemma 5.1.6 then gives that X is strongly even. □

There is a version for more general groups G, communicated to us by Meier.

Definition 5.1.9 (Strongly even G-spectra)**.** Let $X \in \mathbf{Sp}_G$. We say that X is strongly even if the following conditions hold:

(i) We have $\pi^e_{2*-1}(X) = 0$

(ii) For all $e \neq H \leq G$ we have $\pi^H_{*\rho_H - 1}(X) = 0$.

(iii) For all $e \neq H \leq G$ the restriction maps $\mathrm{res}^H_e : \pi^H_{*\rho_H}(X) \to \pi^e_{*|\rho_H|}(X)$ are isomorphisms.

Remark 5.1.10. In the case $G = C_2$ this recovers the Hill–Meier notion of a strongly even C_2-spectrum (Definition 5.1.3).

5.2 Strongly Even Towers of Spectra

In this section we extend the notion of strongly even spectra to the setting of filtered spectra, although for our purposes it is more convenient to think in terms of towers of spectra rather than filtrations.

We use the following conventions. We use a decreasing indexing. A tower $X^\bullet$ of G-spectra consists of a sequence of G-spectra

$$\cdots \longrightarrow X^s \longrightarrow X^{s-1} \longrightarrow \cdots$$

We write $X^\infty = \lim_{s\to\infty} X^s$ for the limit of the tower and think of it as the object the tower aims to compute. The associated graded pieces are defined via the fiber sequences

$$\mathrm{gr}_{s+1} X \longrightarrow X^{s+1} \longrightarrow X^s.$$

Definition 5.2.1 (Strongly even filtered G-spectrum)**.** Let $X^\bullet$ be a tower of G-spectra. We say that $X^\bullet$ is a strongly even tower of G-spectra if the following criteria are met:

(i) The tower is non-negatively indexed.

(ii) The G-spectrum X^0 is a strongly even G-spectrum.

(iii) Each associated graded piece $\mathrm{gr}_s X$ is a strongly even G-spectrum.

Proposition 5.2.2. Let $X^\bullet$ be a strongly even tower of G-spectra. Suppose further that the limit of the underlying non-equivariant tower is even. Let $\varepsilon_H = 1$ if H has even order; $\varepsilon_H = 2$ otherwise. Then, for $H \leq G$, the restriction maps

$$\mathrm{res}^H_e \colon \pi^H_{*\varepsilon_H\rho_H}(X^\infty) \to \pi^e_{*\varepsilon_H|\rho_H|}(X^\infty)$$

are surjections.

Remark 5.2.3. The ε_H appearing in the statement is just a nudge factor to account for the fact ρ_H need not have even dimension.

Proof. The Milnor sequence gives rise to the following map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \lim\limits_s{}^1 \pi^H_{*\varepsilon_H\rho_H+1}(X^s) \arrow[r] \arrow[d, "f"'] & \pi^H_{*\varepsilon_H\rho_H}(X^\infty) \arrow[r] \arrow[d] & \lim\limits_s \pi^H_{*\varepsilon_H\rho_H}(X^s) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \lim\limits_s{}^1 \pi^e_{*\varepsilon_H|\rho_H|+1}(X^s) \arrow[r] & \pi^e_{*\varepsilon_H|\rho_H|}(X^\infty) \arrow[r] & \lim\limits_s \pi^e_{*\varepsilon_H|\rho_H|}(X^s) \arrow[r] & 0.
\end{tikzcd}$$

We wish to show the the middle vertical arrow is surjective. By the Snake Lemma it suffices to prove that f is surjective and that h is an isomorphism.

We begin by noting that X^s is even underlying. Indeed, this is true for X^0 by assumption and so we can argue inductively with the long exact sequence on π_*^e induced by the fiber sequence $\mathrm{gr}_{s+1} X \to X^{s+1} \to X^s$.

- f is a surjection: Since X^s is underlying even, we have $\lim_s^1 \pi^e_{*\varepsilon_H|\rho_H|+1}(X^s) \cong \lim_s^1 0 \cong 0$.
- h is an isomorphism: It suffices to show that

$$\mathrm{res}^H_e \colon \pi^H_{*\varepsilon_H\rho}(X^{s+1}) \to \pi^e_{*\varepsilon_H|\rho_H|}(X^{s+1})$$

is an isomorphism. This is true for $s = -1$ by assumption. We will argue inductively via the map of exact sequences

$$\begin{tikzcd}
\pi^H_{*\varepsilon_H\rho_H+1}(X^s) \arrow[r] \arrow[d] & \pi^H_{*\varepsilon_H\rho_H}(\mathrm{gr}_{s+1}X) \arrow[r] \arrow[d] & \pi^H_{*\varepsilon_H\rho_H}(X^{s+1}) \arrow[r] \arrow[d] & \pi^H_{*\varepsilon_H\rho_H}(X^s) \arrow[r] \arrow[d] & \pi^H_{*\varepsilon_H\rho_H-1}(\mathrm{gr}_{s+1} X) \arrow[d] \\
\pi^e_{*\varepsilon_H|\rho_H|+1}(X^s) \arrow[r] & \pi^e_{*\varepsilon_H|\rho_H|}(\mathrm{gr}_{s+1}X) \arrow[r] & \pi^e_{*\varepsilon_H|\rho_H|}(X^{s+1}) \arrow[r] & \pi^e_{*\varepsilon_H|\rho_H|}(X^s) \arrow[r] & \pi^e_{*\varepsilon_H|\rho_H|-1}(\mathrm{gr}_{s+1} X).
\end{tikzcd}$$

By underlying evenness of X^s the left-most and right-most maps are surjective. By the strong even assumption on the tower, the second map is an isomorphism. By induction hypothesis the fourth map is an isomorphism. So the middle map is an isomorphism by the Five Lemma, as desired.

This finishes the proof. □

Proposition 5.2.4. Let $X^\bullet$ be a strongly even tower of C_2-spectra. Suppose further that the limit of the underlying non-equivariant tower is even. Then, the restriction map

$$\mathrm{res}^{C_2}_e \colon \pi^{C_2}_{*\rho}(X^\infty) \to \pi^e_{2*}(X^\infty)$$

is an isomorphism.

Proof. By assumption $\pi^e_{2*-1}(X^\infty) = 0$. So by Lemma 5.1.8 it suffices to show that $\pi^{C_2}_{*\rho-2}(X^\infty) = 0$. The Milnor sequence gives a short exact sequence

$$0 \longrightarrow \lim_s{}^1 \pi^{C_2}_{*\rho-1}(X^s) \longrightarrow \pi^{C_2}_{*\rho-2}(X^\infty) \longrightarrow \lim_s \pi^{C_2}_{*\rho-2}(X^s) \longrightarrow 0.$$

Thus, to show that $\pi^{C_2}_{*\rho-2}(X^\infty) = 0$ it suffices to show that for all $s \in \mathbb{Z}$ we have $\pi^{C_2}_{*\rho-1}(X^s) = 0$ and $\pi^{C_2}_{*\rho-2}(X^s) = 0$. We will in fact show that each X^s is strongly even. By assumption, X^0 is strongly even, so suppose that X^s is strongly even and we will argue inductively using the fiber sequence

$$\mathrm{gr}_{s+1} X \longrightarrow X^{s+1} \longrightarrow X^s.$$

Thus, the sequence

$$\pi^{C_2}_V(\mathrm{gr}_{s+1} X) \longrightarrow \pi^{C_2}_V(X^{s+1}) \longrightarrow \pi^{C_2}_V(X^s)$$

is exact for every $V \in \mathrm{RO}(C_2)$. Since $\mathrm{gr}_{s+1} X$ and X^s are strongly even, we deduce with Greenlees' gap characterization (Lemma 5.1.6) that X^{s+1} is strongly even. □

5.3 Cohomological Slice Tower

The goal of this section is to set up a tower of spectra, thereby reducing computations of $\underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, E)$ to Bredon cohomology computations. We do this by applying the equivariant slice tower to E.

The equivariant slice tower was introduced by Hill–Hopkins–Ravenel and was central to their solution of the Kervaire invariant one-problem [HHR16]. For the group C_2, the C_2-slice tower was constructed and first studied by Dugger in [Dug05]. We only recall the minimal input we need to set up a cohomological version of the slice tower. For a more complete treatment the standard references are [HHR16, Hil12, Ull13]. For an introduction we recommend and have benefited from [Gui22]. See also [AKKQ25, Section 2.1] for a concise treatment in the same language we use.

Remark 5.3.1. The formal properties of the slice tower were refined by Ullman [Ull13] in his construction of the *regular slice tower*. We will use Ullman's regular slice tower but for brevity sake only say slice tower.

Example 5.3.2. The slice tower often organizes information in a more efficient way. The main motivating example to consider is Atiyah's K-theory with Reality $\mathrm{ku}_{\mathbb{R}}$. The equivariant Bott map for $\mathrm{ku}_{\mathbb{R}}$ sits in the cofiber sequence

$$S^{\rho} \otimes \mathrm{ku}_{\mathbb{R}} \xrightarrow{\overline{\beta}} \mathrm{ku}_{\mathbb{R}} \longrightarrow \mathrm{H}\underline{\mathbb{Z}}$$

Since the equivariant Bott map shifts degree by ρ, equivariant Bott periodicity appears complicated in the equivariant Postnikov tower. It is better visible in the slice tower. By [Dug05], the slice tower for $\mathrm{ku}_{\mathbb{R}}$ is given as follows:

$$\begin{tikzcd}
 & S^{2\rho} \otimes \mathrm{H}\underline{\mathbb{Z}} \arrow[d] & 0 \arrow[d] & S^{\rho} \otimes \mathrm{H}\underline{\mathbb{Z}} \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P^{\leq 4}\,\mathrm{ku}_{\mathbb{R}} \arrow[r] & P^{\leq 3}\,\mathrm{ku}_{\mathbb{R}} \arrow[r] & P^{\leq 2}\,\mathrm{ku}_{\mathbb{R}} \arrow[r] & P^{\leq 1}\,\mathrm{ku}_{\mathbb{R}} \arrow[r] & P^{\leq 0}\,\mathrm{ku}_{\mathbb{R}} \simeq \mathrm{H}\underline{\mathbb{Z}}
\end{tikzcd}$$

The slice tower $P^{\leq \bullet}\,\mathrm{ku}_{\mathbb{R}}$ is depicted as the horizontal tower of C_2-spectra. The vertical fibers are the associated graded pieces of the slice tower. The n'th associated graded piece is called the n'th *slice* of $\mathrm{ku}_{\mathbb{R}}$. This is exactly the analogue of the Postnikov tower for ku with $S^{2n} \otimes \mathrm{H}\mathbb{Z}$ replaced by $S^{n\rho} \otimes \mathrm{H}\underline{\mathbb{Z}}$.

Lemma 5.3.3. Let X be a strongly even C_2-spectrum. The odd slices $P^{2n-1}_{2n-1}X$ vanish and the even slices are given by $P^{2n}_{2n}X \simeq \Sigma^{n\rho}\mathrm{H}\underline{\pi}_{n\rho}X$.

Proof. As summarized in [HM17, Proposition 2.13] this follows from [HHR09, Proposition 4.20, Lemma 4.23] and [Hil12, Corollary 2.16]. □

Now we are ready to set up a cohomological version of the slice tower. This is a cohomological version of the tower constructed by Carrick–Hill–Ravenel in their study of the homological slice spectral sequence [CHR24].

Construction 5.3.4 (Cohomological slice tower)**.** Let $X, E \in \mathbf{Sp}_G$. We define the cohomological slice tower for X and E as the tower of G-spectra given by

$$\mathrm{CST}(X; E) := \underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, P^{\leq \bullet}E).$$

Note that the s'th associated graded is given by

$$\mathrm{gr}_s\,\mathrm{CST}(X; E) = \underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, P^s_s E).$$

Applying $\underline{\pi}_V$ this gives rise to a spectral sequence of Mackey functors of the form

$$E^1_{V,s} = \underline{\pi}_V(\underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, P^s_s E)) \implies \underline{\pi}_V(\underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, E)).$$

Remark 5.3.5.

(i) In general, convergence of the associated spectral sequence is subtle. As such, instead argue directly with the tower.

(ii) In all our applications of interest E will be connective. So the cohomological slice tower takes the form

$$\cdots \longrightarrow \underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, P^{\leq s}E) \longrightarrow \cdots \longrightarrow \underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, P^{\leq 0}E) \longrightarrow 0.$$

In particular, $\mathrm{CST}(X; E)$ is positively indexed for connective E.

(iii) Since the slice tower $P^{\leq \bullet}E$ is exhaustive, also $\mathrm{CST}(X; E)$ is exhaustive.

(iv) By construction, the underlying non-equivariant tower of the cohomlogical slice tower $\mathrm{CST}(X; E)$ is the tower whose associated spectral sequence is the usual cohomological Atiyah–Hirzebruch spectral sequence for X^e and E^e.

By Lemma 5.3.3, computing the associated graded of the cohomological slice tower involves computing Bredon cohomology. In general, such computations can be very difficult. The Bredon cohomology counterparts of many classical cohomology computations are not known. However, Hill has identified a large class of spectra for which Bredon cohomology computations are largely formal [Hil22a]. We finish this section by leveraging the techniques of Hill to give criterion to check when the cohomological slice tower is strongly even.

The following definition isolates the kinds of G-spectra we are interested in computing Bredon cohomology of.

Definition 5.3.6. Let $X \in \mathbf{Sp}_G$. We say that X has finite type free homology if there is an equivalence $\mathrm{H}\underline{\mathbb{Z}} \otimes X \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{i \in I} S^{n_i \rho_G}$ for some indexing set I such that each dimension appears only finitely many times in the sum.

Since we make use of the results of Hill's freeness arguments repeatedly, we review the main idea. We are grateful to Christian Carrick for explaining this to us.

Lemma 5.3.7 (Hill's freeness argument)**.** Let $X \in \mathbf{Sp}_G$. Suppose that X has finite type free homology with $\mathrm{H}\underline{\mathbb{Z}} \otimes X \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{i \in I} S^{n_i \rho_G}$. Let $M \in \mathbf{Ab}$ and consider the associated constant Mackey functor $\underline{M}$. Then, there is an equivalence

$$\underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, \mathrm{H}\underline{M}) \simeq \mathrm{H}\underline{M} \otimes \bigoplus_{i \in I} S^{-n_i \rho_G}.$$

Proof. This follows from the much more general result of [Hil22a, Theorem 3.38]; for the benefit of the reader we exhibit the details from the proof [Hil22a, Theorem 3.38] to prove the specific case we need directly.

Since $\underline{M}$ is a constant Mackey functor, $\mathrm{H}\underline{M}$ is a $\mathrm{H}\underline{\mathbb{Z}}$-module. We then have the following equivalences:

$$\begin{aligned}
\underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, \mathrm{H}\underline{M}) &\simeq \underline{\mathrm{map}}_{\underline{\mathbf{LMod}}_{\mathrm{H}\underline{\mathbb{Z}}}}(\mathrm{H}\underline{\mathbb{Z}} \otimes X, \mathrm{H}\underline{M}) \\
&\simeq \underline{\mathrm{map}}_{\underline{\mathbf{LMod}}_{\mathrm{H}\underline{\mathbb{Z}}}}\left(\mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{i\in I} S^{n_i\rho_G}, \mathrm{H}\underline{M}\right) \\
&\simeq \underline{\mathrm{map}}_{\mathbf{Sp}^G}\left(\mathbb{S} \otimes \bigoplus_{i\in I} S^{n_i\rho_G}, \mathrm{H}\underline{M}\right) \\
&\simeq \underline{\mathrm{map}}_{\mathbf{Sp}^G}\left(\bigoplus_{i\in I} \mathbb{S}^{n_i\rho_G}, \mathrm{H}\underline{M}\right) \\
&\simeq \prod_{i\in I} \underline{\mathrm{map}}_{\mathbf{Sp}^G}\left(\mathbb{S}^{n_i\rho_G}, \mathrm{H}\underline{M}\right) \\
&\simeq \prod_{i\in I} \left(\mathrm{H}\underline{M} \otimes S^{-n_i\rho_G}\right) \\
&\simeq \mathrm{H}\underline{M} \otimes \bigoplus_{i\in I} S^{-n_i\rho_G}
\end{aligned}$$

The finite type hypothesis is invoked for the final equivalence. □

Combining Hill's freeness argument with the results on towers of strongly even spectra, we can deduce when the cohomological slice tower is strongly even. For clarity, we treat the case $G = C_2$ separately.

Proposition 5.3.8. Let $E \in \mathbf{Sp}_{C_2}$ be strongly even and connective. Suppose that $X \in \mathbf{Sp}_{C_2}$ has finite type free homology with $\mathrm{H}\underline{\mathbb{Z}} \otimes X \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{i\in I} S^{n_i\rho}$. Then the restriction map

$$\mathrm{res}^{C_2}_e \colon \pi^{C_2}_{*\rho}(\underline{\mathrm{map}}_{\mathbf{Sp}^{C_2}}(X, E)) \to \pi_{2*}(\mathrm{map}_{\mathbf{Sp}^{C_2}}(X, E))$$

is an equivalence.

Proof. Consider the cohomological slice tower $\mathrm{CST}(X; E)$. We will use the conditions on X and E to show that the cohomological slice tower is a strongly even tower of C_2-spectra, and that the underlying spectrum $\mathrm{map}_{\mathbf{Sp}}(X^e, E^e)$ is even. The result then follows from Proposition 5.2.4.

First, we handle the underlying statement. Since E is strongly even, its underlying spectrum is even. Similarly, the assumption on X shows that its underlying homology is finite type and is given by $\mathrm{H}\mathbb{Z} \otimes X \simeq \mathrm{H}\mathbb{Z} \otimes \bigoplus_{i\in I} S^{2n_i}$. The usual Atiyah–Hirzebruch spectral sequence then shows that $\mathrm{map}_{\mathbf{Sp}}(X^e, E^e)$ is even.

It remains to see that the associated graded pieces of $\mathrm{CST}(X; E)$ are strongly even, that is, we must show that $\underline{\mathrm{map}}_{\mathbf{Sp}^{C_2}}(X, P^s_s E)$ is strongly even. Since E is strongly even, its odd slices vanish, and its even slices are given by $P^{2s}_{2s} E = \Sigma^{s\rho}\mathrm{H}\underline{\pi}_{s\rho}(E)$, as recalled in Lemma 5.3.3. Strong evenness is preserved under $s\rho$ suspensions, it suffices to show that $\underline{\mathrm{map}}_{\mathbf{Sp}^{C_2}}(X, \mathrm{H}\underline{\pi}_{s\rho}(E))$ is strongly even. Since E is strongly even, $\underline{\pi}_{s\rho}(E)$ is a constant Mackey functor. So the result follows from Lemma 5.3.7. □

The case for general G is more subtle. Suppose that $X \in \mathbf{Sp}_G$ has finite type free homology with

$$\mathrm{H}\underline{\mathbb{Z}} \otimes X \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{i\in I} S^{n_i\rho_G}$$

for an indexing set I. For any constant Mackey functor $\underline{M}$ we have

$$\underline{\mathrm{map}}_{\mathbf{Sp}^G}(X, \mathrm{H}\underline{M}) \simeq \mathrm{H}\underline{M} \otimes \bigoplus_{i\in I} S^{-k_i\rho_G}$$

by Hill freeness (Lemma 5.3.7). Mimicking our proof in the previous proposition (Proposition 5.3.8) we need to check strong freeness of a G-spectrum of this form.

Since ρ_G-suspensions of strongly even G-spectra are strongly even, it suffices to show that $\mathrm{H}\underline{M}$ is strongly even. Using cell structures on $S^{n\rho_G}$, an argument with Bredon homology for $n > 0$, and Bredon cohomology for $n < 0$, shows that $\pi^G_{*\rho_G}(\mathrm{H}\underline{M}) = 0$ for $* \neq 0$. By assumption on $\underline{M}$, the restriction maps in $\underline{\pi}_0(\mathrm{H}\underline{M})$ are isomorphisms. The last remaining condition of strong evenness is that $\pi^H_{*\rho_H - 1}(\mathrm{H}\underline{M})$ should vanish for $e \neq H \leq G$.

For general G, we do not know a reasonable condition that implies this. However, in the case that $G = C_{p^k}$, these is a condition in [Hil22a, Corollary 4.10].

Proposition 5.3.9. Let $G = C_{p^k}$ for some prime p and some $k \in \mathbb{N}$. Let $X \in \mathbf{Sp}_G$ and let $E \in \mathbf{Sp}_G$ be strongly even and connective. Suppose the following:

(i) The G-spectrum X has finite type free homology with $\mathrm{H}\underline{\mathbb{Z}} \otimes X \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{i \in I} \mathbb{S}^{n_i \rho}$ for some indexing set I.

(ii) Each $P^{n|\rho_G|}_{n|\rho_G|} E$ is an $n\rho_G$-fold suspension of an Eilenberg–MacLane spectrum for a constant Mackey functor, and the other remaining slices are contractible.

Then, the cohomological slice tower for X and E is strongly even. In particular, it follows that the restriction map

$$\pi^G_{*\varepsilon_G \rho_G}(\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^G}(X, E)) \longrightarrow \pi^e_{*\varepsilon_G |\rho_G|}(\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^G}(X, E))$$

is a surjection; here $\varepsilon_G = 1$ if G has even order and $\varepsilon_G = 2$ otherwise (Remark 5.2.3).

Proof. The assumptions and argument above show that $\underline{\mathrm{map}}_{\underline{\mathbf{Sp}}^G}(X, P^s_s E)$ is strongly even by [Hil22a, Corollary 4.10]. So the result follows from Proposition 5.2.2 by arguing exactly as in Proposition 5.3.8. □

6 Obstruction Theory for Structured Orientations & Splittings of Thom Spectra

In this section we isolate arguments that would otherwise be repeated several times. We hope this is also a benefit for the reader with different applications in mind. For readability, we do not state these results in full generality, but our proofs suggest straightforward generalizations.

For future work, we require these results in the level of generality of R-module Thom spectra; the reader only interested in Theorem H and Theorem E loses nothing by assuming $R = \mathbb{S}$. For simplicity, we will assume that R is a $\mathbb{E}^G_\infty$-ring, but this is not strictly needed.

The arguments in this section are generalizations of the main argument in the sketch proof of Theorem H from the beginning of Part II. The main idea of this section can be summarized as follows: It is often easier to lift a non-equivariant $\mathbb{E}_{\dim(V)}$-map $\mathrm{M}f^e \to E^e$ to an equivariant $\mathbb{E}_V$-map $\mathrm{M}f \to E$, than it is to lift an equivariant but non-multiplicative map $\mathrm{M}f \to E$ to an $\mathbb{E}_V$-map.

6.1 Lifting Orientations

For historical reasons, and because we mainly have MU in mind, in this section we will write $\mathrm{M}f$ instead of $\mathrm{Th}(f)$ for Thom spectra.

Proposition 6.1.1. Let $R \in \mathbf{Alg}_{\mathbb{E}_\infty^G}(\underline{\mathbf{Sp}}_G)$ and $E \in \mathbf{Alg}_{\mathbb{E}_\infty^G}(\underline{\mathbf{LMod}}_R^G)$. Let V be a G-representation and let X be an V-loop space with a map $f\colon X \to \underline{\mathrm{Pic}}_G(R)$ in $\mathbf{Alg}_{\mathbb{E}_{n\rho_G}}(\underline{\mathcal{S}}_G)$. Suppose there is an $\mathbb{E}_V$-R-algebra map $\mathrm{M}f \to E$. Then there is an equivalence

$$\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_V}(\underline{\mathbf{LMod}}_R^G)}(\mathrm{M}f, E) \simeq \Omega^\infty \underline{\mathrm{map}}_{\mathbf{Sp}^G}(\Sigma^\infty \mathrm{B}^V X, \Sigma^V \mathrm{gl}_1(E)).$$

in $\mathcal{S}_*^G$.

Proof. We perform the following string of equivalences:

$$\begin{aligned}
\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho_G}}(\underline{\mathbf{LMod}}_R^G)}(\mathrm{M}f, E) &\simeq \underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho_G}}(\underline{\mathbf{LMod}}_R^G)}(R \otimes \Sigma_+^\infty X, E) \\
&\simeq \underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho_G}}(\underline{\mathbf{Sp}}_G)}(\Sigma_+^\infty X, E) \\
&\simeq \underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho_G}}(\underline{\mathcal{S}}_G)}(X, \Omega^\infty E) \\
&\simeq \underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathbb{E}_{n\rho_G}}(\underline{\mathcal{S}}_G)}(X, \mathrm{GL}_1(E)) \\
&\simeq \underline{\mathrm{Map}}_{\mathcal{S}_*^G}(\mathrm{B}^V X, \mathrm{B}^V \mathrm{GL}_1(E)) \\
&\simeq \underline{\mathrm{Map}}_{\mathbf{Sp}^G}(\Sigma^\infty \mathrm{B}^V X, \Sigma^V \mathrm{gl}_1(E)) \\
&\simeq \Omega^\infty \underline{\mathrm{map}}_{\mathbf{Sp}^G}(\Sigma^\infty \mathrm{B}^V X, \Sigma^V \mathrm{gl}_1(E))
\end{aligned}$$

In order the equivalences are justified as follows: the first equivalence uses the multiplicative Thom isomorphism Corollary 4.2.10 (ii); the second uses the free R-module adjunction; the third uses the $\Sigma_+^\infty \dashv \Omega^\infty$ adjunction; the fourth uses Corollary 3.4.12 along with the assumption that X is grouplike; the fifth and sixth both use Theorem 3.4.3; the final is by definition of $\underline{\mathrm{map}}_{\mathbf{Sp}^G}$. □

Corollary 6.1.2. Let $R \in \mathbf{Alg}_{\mathbb{E}_\infty^G}(\underline{\mathbf{Sp}}_G)$ and $E \in \mathbf{Alg}_{\mathbb{E}_\infty^G}(\underline{\mathbf{LMod}}_R^G)$. Let $n \in \mathbb{N}$ and let X be an $n\rho_G$-loop space with a map $f\colon X \to \underline{\mathrm{Pic}}_G(R)$ in $\mathbf{Alg}_{\mathbb{E}_{n\rho_G}}(\underline{\mathcal{S}}_G)$. Suppose there is an $\mathbb{E}_{n\rho_G}$-R-algebra map $\mathrm{M}f \to E$. Further suppose the following:

(i) The cohomological slice tower $\mathrm{CST}(\Sigma^\infty \mathrm{B}^{n\rho_G} X; \Sigma^{n\rho_G}\mathrm{gl}_1 E)$ is strongly even.

(ii) The underlying non-equivariant spectrum of $\underline{\mathrm{map}}_{\mathbf{Sp}^G}(\Sigma^\infty \mathrm{B}^{n\rho_G} X, \Sigma^{n\rho_G}\mathrm{gl}_1(E))$ is even.

Then, the restriction map

$$\mathrm{res}_e^G\colon \pi^G_{*\varepsilon_G\rho_G}\left(\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho_G}}(\underline{\mathbf{LMod}}_R^G)}(\mathrm{M}f, E)\right) \to \pi^e_{*\varepsilon_G|\rho_G|}\left(\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho_G}}(\underline{\mathbf{LMod}}_R^G)}(\mathrm{M}f, E)\right)$$

is surjective. When $G = C_2$, it is furthermore an isomorphism.

Proof. By Proposition 6.1.1, the general case follows from Proposition 5.2.2. The fact that the map is an isomorphism for C_2 follows from Proposition 5.2.4. □

Remark 6.1.3. The strong multiplicative structures on R and E were assumed just so that $\mathrm{GL}_1(E)$ can be delooped to $\mathrm{gl}_1(E)$. Suppose we only have the following weaker assumptions: R is an $\mathbb{E}_V \otimes \mathbb{E}_2$-ring; E is an $\mathbb{E}_V \otimes \mathbb{E}_1$-ring, and there is an $\mathbb{E}_V \otimes \mathbb{E}_1$-ring map $R \to E$; and E admits an $\mathbb{E}_V$-$\mathrm{M}f$-orientation. Then, the first five equivalences in the proof of Proposition 6.1.1 hold and give an equivalence

$$\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_V}(\underline{\mathbf{LMod}}_R^G)}(\mathrm{M}f, E) \simeq \underline{\mathrm{Map}}_{\mathcal{S}_*^G}(\mathrm{B}^V X, \mathrm{B}^V \mathrm{GL}_1(E)).$$

in $\mathcal{S}_G$.

For the sake concreteness sake we restrict to $G = C_2$ for the following.

Theorem 6.1.4. Let $R \in \mathbf{Alg}_{\mathbb{E}_\infty^{C_2}}(\underline{\mathbf{Sp}}_{C_2})$ and $E \in \mathbf{Alg}_{\mathbb{E}_\infty^{C_2}}(\underline{\mathbf{LMod}}_R^{C_2})$. Let $n \in \mathbb{N}$ and let X be an $n\rho$-loop space with a map $f\colon X \to \underline{\mathrm{Pic}}_{C_2}(R)$ in $\mathbf{Alg}_{\mathbb{E}_{n\rho}}(\underline{\mathcal{S}}_{C_2})$. Suppose that there exists an $\mathbb{E}_{n\rho}$-R-algebra map $\mathrm{M}f \to E$. Furthermore, suppose that $\mathrm{B}^{n\rho}X$ has finite type free homology with $\mathrm{H}\underline{\mathbb{Z}} \otimes \mathrm{B}^{n\rho}X \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{i\in I} S^{n_i\rho}$ for some indexing set I. Then, the restriction map

$$\mathrm{res}_e^{C_2}\colon \pi_{*\rho}^{C_2}\left(\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})}(\mathrm{M}f, E)\right) \to \pi_{*|\rho|}^{e}\left(\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})}(\mathrm{M}f, E)\right)$$

is an isomorphism.

Proof. By Proposition 6.1.1 we have an equivalence of pointed C_2-spaces

$$\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})}(\mathrm{M}f, E) \simeq \Omega^\infty \underline{\mathrm{map}}_{\mathbf{Sp}^{C_2}}(\Sigma^\infty \mathrm{B}^{n\rho}X, \Sigma^{n\rho}\mathrm{gl}_1(E)).$$

By the assumptions on $\mathrm{B}^{n\rho}X$ and $\Sigma^{n\rho}\mathrm{gl}_1(E)$, the statement follows from Proposition 5.3.8. Note here that strong evenness of E implies strong evenness of $\Sigma^{n\rho}\mathrm{gl}_1(E)$. Moreover, $\Sigma^{n\rho}\mathrm{gl}_1(E)$ is connective by construction. $\square$

Remark 6.1.5. Using Proposition 5.3.9, a similar statement can be made in the case $G = C_{p^k}$, although in this case the restriction map is usually only surjective. With further assumptions similar statements can be made for general finite G, again the restriction map is usually only surjective.

6.2 Lifting Idempotent Splittings

The Thom isomorphism (Theorem 4.2.9) is crucial for many of our arguments and this requires the existence of an orientation. Observe the following obvious but useful orientation of Thom spectra. If $\mathrm{M}f$ is an $\mathbb{E}_V$-R-Thom spectrum, then $\mathrm{id}_{\mathrm{M}f}\colon \mathrm{M}f \to \mathrm{M}f$ is in particular an $\mathbb{E}_V$-R-algebra map. This is an orientation if f starts from a grouplike space (Example 4.2.8). We will abuse this fact to study lifts of idempotents.

For simplicity, the following is not stated in full generality.

Theorem 6.2.1. Let $R \in \mathbf{Alg}_{\mathbb{E}_\infty^{G}}(\underline{\mathbf{Sp}}_{C_2})$ and X be an infinite loop C_2-space. Let $f\colon X \to \underline{\mathrm{Pic}}_{C_2}(R)$ be a map in $\mathbf{Alg}_{\mathbb{E}_\infty^{C_2}}(\underline{\mathcal{S}}_{C_2})$ and suppose that $\mathrm{M}f$ is strongly even. Suppose that $\mathrm{B}^{n\rho}X$ has finite type free homology with

$$\mathrm{H}\underline{\mathbb{Z}} \otimes \mathrm{B}^{n\rho}X \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{i\in I} S^{n_i\rho}$$

for some indexing set I. Then, any $\mathbb{E}_{2n}$-R^e-algebra idempotent on the underlying $g\colon \mathrm{M}f^e \to \mathrm{M}f^e$ lifts to an $\mathbb{E}_{n\rho}$-R-algebra idempotent $\overline{g}\colon \mathrm{M}f \to \mathrm{M}f$.

Remark 6.2.2. We momentarily defer the proof to bring attention to a potential cause of contention. Namely, the subtly that an idempotent in an ∞-category is not simply the data of a map e along with a homotopy witness $e \circ e \simeq e$. Such data only says that e is a homotopy idempotent. Compare the definition of a homotopy idempotent [Lur18, Tag 041W] to that of an idempotent [Lur18, Tag 041B]. However, in a stable ∞-category a homotopy idempotent uniquely extends to an idempotent, see [Lur17, Warning 1.2.4.8]. On the other hand, algebras in a stable category are hardly ever stable, but [Lur18, Tag 0420] can be used as a workaround.

Proof of Theorem 6.2.1. By Theorem 6.1.4, the conditions on $\mathrm{M}f$, and $\mathrm{B}^V X$ imply that the restriction map

$$\mathrm{res}_e^{C_2}\colon \pi_0^{C_2}\left(\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})}(\mathrm{M}f, \mathrm{M}f)\right) \to \pi_0^{e}\left(\underline{\mathrm{Map}}_{\underline{\mathbf{Alg}}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})}(\mathrm{M}f, \mathrm{M}f)\right)$$

is an isomorphism. So there is an $\mathbb{E}_{n\rho}$-R-algebra map $\overline{g}\colon \mathrm{M}f \to \mathrm{M}f$ lifting g. Since the restriction map is an isomorphism it follows that $\overline{g}$ is a *homotopy* idempotent in the $\mathbf{Alg}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})$. It remains to see that this is an actual idempotent.

First note that by forgetting the multiplicative structure that $\overline{g}$ defines a homotopy idempotent in $\mathbf{LMod}_R^{C_2}(\underline{\mathbf{Sp}}_{C_2})$. By [Lur17, Lemma 1.2.4.6, Warning 1.2.4.8] $\overline{g}$ then defines an idempotent in $\mathbf{LMod}_R^{C_2}(\underline{\mathbf{Sp}}_{C_2})$. Since limits and filtered colimits in $\mathbf{Alg}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})$ are computed in $\mathbf{LMod}_R^{C_2}(\underline{\mathbf{Sp}}_{C_2})$, it follows from [Lur18, Tag 0420] that the map $\overline{g}$ is an idempotent in $\mathbf{Alg}_{\mathbb{E}_{n\rho}}(\underline{\mathbf{LMod}}_R^{C_2})$. □

Remark 6.2.3. We restricted to $G = C_2$ to ensure that the restriction map is an isomorphism. For $G \neq C_2$, if one happens to know that the restriction map is an isomorphism, then everything after the first sentence of the proof goes through as expected.

7 Real Orientations & Multiplication on $\mathrm{BP}_{\mathbb{R}}$

7.1 Constructing Multiplicative Real Orientations

Let E be a strongly even $\mathbb{E}_\infty^{C_2}$-ring spectrum. The goal of this section is to show E admits an $\mathbb{E}_\rho$-$\mathrm{MU}_{\mathbb{R}}$-orientation. Informally speaking, our strategy is to produce an orientation via a versal example. More precisely, we will construct an $\mathbb{E}_\rho$-map $\mathrm{MU}_{\mathbb{R}} \to \mathrm{MW}_{\mathbb{R}}$, where $\mathrm{MW}_{\mathbb{R}}$ is a $\mathbb{E}_\infty^{C_2}$-ring spectrum constructed by Angelini-Knoll–Kong–Quigley which admits $\mathbb{E}_\infty^{C_2}$-ring maps $\mathrm{MW}_{\mathbb{R}} \to E$ [AKKQ25]. This is a Real version of the spectrum MW considered by Hahn–Raksit–Wilson [HRW24, Definition 3.2.9].

Construction 7.1.1 ([AKKQ25, 5.8])**.** Let $\overline{\gamma}\colon \mathrm{MU}_{\mathbb{R}} \to \mathrm{ku}_{\mathbb{R}}$ be any map of C_2-spectra which is surjective on $\pi_{*\rho}^{C_2}$. For example we may take Hahn–Shi's lift of the Conner–Floyd orientation [HS20]. Applying $\Omega^\infty\Sigma^\rho$, using equivariant Bott perodicity (Example 3.4.5), and postcomposing with the Real J-homomorphism (Section A.2) gives an $\mathbb{E}_\infty^{C_2}$-map

$$\Omega^\infty\Sigma^\rho \mathrm{MU}_{\mathbb{R}} \to \Omega^\infty\Sigma^\rho\, \mathrm{ku}_{\mathbb{R}} \simeq \mathrm{BU}_{\mathbb{R}} \to \underline{\mathrm{Pic}}_{C_2}(\underline{\mathbf{Sp}}^{C_2}).$$

The spectrum $\mathrm{MW}_{\mathbb{R}}$ is defined to the Thom spectrum of this map.

Remark 7.1.2 (Real Wilson Spaces)**.**

(i) The construction of $\mathrm{MW}_{\mathbb{R}}$ by Angelini-Knoll–Kong–Quigley is a (Real) equivariant refinement of a construction of Hahn–Raksit–Wilson. In their work on the even filtration [HRW24, Definition 3.2.9], Hahn–Raksit–Wilson constructed MW as the Thom spectrum of

$$\gamma\colon \Omega^\infty\Sigma^2\, \mathrm{MU} \to \Omega^\infty\Sigma^2\, \mathrm{ku} \simeq \mathrm{BU} \to \mathrm{Pic}(\mathbf{Sp}).$$

By construction, Hahn–Raksit–Wilson's MW is the underlying non-equivariant spectrum of Angelini-Knoll–Kong–Quigley's $\mathrm{MW}_{\mathbb{R}}$.

(ii) The base space of $\mathrm{MW}_{\mathbb{R}}$ is an example of a *Real Wilson Space*. The non-equivariant version was first extensively studied by Wilson in his PhD thesis [Wil73, Wil75], hence the nomenclature. The study of the Real version was undertaken by Hill–Hopkins [HH18]. In particular, they show that $\Omega^\infty\Sigma^{n\rho}\mathrm{MU}_{\mathbb{R}}$ has finite type free homology with

$$\mathrm{H}\underline{\mathbb{Z}} \otimes \Omega^\infty\Sigma^{n\rho}\mathrm{MU}_{\mathbb{R}} \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{i\in I} S^{n_i\rho},$$

for a certain indexing set I and $n_i \in \mathbb{N}$, see [HH18, Theorem 1.2]. More precisely, the computation of Hill–Hopkins shows that $\Omega^\infty\Sigma^{n\rho}\mathrm{MU}_{\mathbb{R}}$ has free $\mathrm{H}\underline{\mathbb{Z}}$-homology, and that

$\mathrm{H}\underline{\mathbb{Z}} \otimes \Omega^\infty \Sigma^{n\rho} \mathrm{MU}_\mathbb{R}$ is strongly even. The finite type assumption follows from the non-equivariant computation Wilson [Wil73, Theorem 3.3, Corollary 3.4].

Note that the work of Hill–Hopkins also computes the $\underline{\mathbb{Z}}$-homology of every $m\rho$-fold delooping of $\Omega^\infty\Sigma^\rho \mathrm{MU}_\mathbb{R}$. By Lemma 3.4.4 we have

$$\mathrm{B}^{m\rho}\Omega^\infty\Sigma^{n\rho}\mathrm{MU}_\mathbb{R} \simeq \Omega^\infty\Sigma^{(n+m)\rho}\mathrm{MU}_\mathbb{R}.$$

So for every m, $\mathrm{B}^{m\rho}\Omega^\infty\Sigma^{n\rho}\mathrm{MU}_\mathbb{R}$ has finite type free homology with

$$\mathrm{H}\underline{\mathbb{Z}} \otimes \mathrm{B}^{m\rho}\Omega^\infty\Sigma^{\rho}\mathrm{MU}_\mathbb{R} \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{j\in J} S^{n_j\rho}.$$

for a certain indexing set J.

One of the main motivations for the construction of $\mathrm{MW}_\mathbb{R}$ is that it provides multiplicative orientations for many rings of interest.

Theorem 7.1.3 ([AKKQ25])**.** Let E be a strongly even $\mathbb{E}_\infty^{C_2}$-ring spectrum. There exists a $\mathbb{E}_\infty^{C_2}$-ring map $\mathrm{MW}_\mathbb{R} \to E$.

Proof. This follows from [AKKQ25, Theorem 5.10 (1)]. Specifically, see the last paragraph of the proof. □

Proposition 7.1.4 ($\mathrm{MW}_\mathbb{R}$-splitting)**.** There is an $\mathbb{E}_\rho$-idempotent of $\mathrm{MW}_\mathbb{R}$ that splits off $\mathrm{MU}_\mathbb{R}$ as an $\mathbb{E}_\rho$-retract. In particular, there is an $\mathbb{E}_\rho$-map $\mathrm{MU}_\mathbb{R} \to \mathrm{MW}_\mathbb{R}$.

Proof. There are three steps: First we discuss a non-equivariant $\mathbb{E}_2$-splitting after Hahn–Raksit–Wilson. Then, we lift the non-equivariant splitting to an equivariant splitting. Finally, we check that the summand split off equivariantly is indeed $\mathrm{MU}_\mathbb{R}$.

Hahn–Raksit–Wilson show $\mathrm{MW} \simeq \mathrm{MU} \otimes \Sigma^\infty_+ \mathrm{fib}(\Omega^\infty\Sigma^2\gamma)$ as $\mathbb{E}_2$-rings in the proof of [HRW24, Theorem 3.2.10]. Now consider the map $\mathrm{fib}(\Omega^\infty\Sigma^2\gamma) \to *$ which makes $\Sigma^\infty_+\mathrm{fib}(\Omega^\infty\Sigma^2\gamma)$ into an augmented $\mathbb{E}_2$-algebra. Using the augmentation we can construct an $\mathbb{E}_2$-idempotent

$$e\colon\ \mathrm{MW} \simeq \mathrm{MU} \otimes \Sigma^\infty_+\mathrm{fib}(\Omega^\infty\Sigma^2\gamma) \to \mathrm{MU}\otimes \mathbb{S} \to \mathrm{MU} \otimes \Sigma^\infty_+\mathrm{fib}(\Omega^\infty\Sigma^2\gamma) \simeq \mathrm{MW}$$

which splits off MU from MW. Thus,

$$\mathrm{MU} \simeq \mathrm{colim}\left(\mathrm{MW} \xrightarrow{e} \mathrm{MW} \xrightarrow{e} \mathrm{MW} \xrightarrow{e} \cdots\right).$$

This completes the first step.

Now, we show that e can be lifted to an $\mathbb{E}_\rho$-idempotent $\bar{e}\colon \mathrm{MW}_\mathbb{R} \to \mathrm{MW}_\mathbb{R}$. So we must check the conditions of Theorem 6.2.1. The fact that $\mathrm{MW}_\mathbb{R}$ is strongly even is [AKKQ25, Theorem C, Proposition 5.9.]. It remains to check the condition on the base space $\Omega^\infty\Sigma^\rho\mathrm{MU}_\mathbb{R}$ of $\mathrm{MW}_\mathbb{R}$. To apply Theorem 6.2.1 in the case of $\mathbb{E}_\rho$-structures, we must show that $\mathrm{B}^\rho\Omega^\infty\Sigma^\rho\mathrm{MU}_\mathbb{R}$ has finite type free homology with

$$\mathrm{H}\underline{\mathbb{Z}} \otimes \mathrm{B}^{\rho}\Omega^\infty\Sigma^{\rho}\mathrm{MU}_\mathbb{R} \simeq \mathrm{H}\underline{\mathbb{Z}} \otimes \bigoplus_{n_i\in I} S^{n_i\rho}$$

This follows from the work of Hill–Hopkins as recalled in Remark 7.1.2. So Theorem 6.2.1 produces an $\mathbb{E}_\rho$-idempotent $\bar{e}\colon \mathrm{MW}_\mathbb{R} \to \mathrm{MW}_\mathbb{R}$ lifting e.

It remains to check that the $\mathbb{E}_\rho$-ring split off is indeed $\mathrm{MU}_\mathbb{R}$. We temporarily make the distinction and write

$$\overline{\mathrm{MU}}_\mathbb{R} \coloneqq \mathrm{colim}\left(\mathrm{MW}_\mathbb{R} \xrightarrow{\bar{e}} \mathrm{MW}_\mathbb{R} \xrightarrow{\bar{e}} \mathrm{MW}_\mathbb{R} \xrightarrow{\bar{e}} \cdots\right).$$

As $\overline{\mathrm{MU}}_{\mathbb{R}}$ is a filtered colimit of strongly even C_2-spectra, by Greenlees' gap characterization of strong evenness (Lemma 5.1.6), it is immediate that $\overline{\mathrm{MU}}_{\mathbb{R}}$ is strongly even. Thus, by [HM17, Lemma 3.4], it suffices to produce a map $\overline{\mathrm{MU}}_{\mathbb{R}} \to \mathrm{MU}_{\mathbb{R}}$ that is a non-equivariant equivalence. By construction of $\overline{\mathrm{MU}}_{\mathbb{R}}$ we have a map $\overline{\mathrm{MU}}_{\mathbb{R}} \to \mathrm{MW}_{\mathbb{R}}$ whose underlying non-equivariant map is the inclusion $\mathrm{MU} \hookrightarrow \mathrm{MW}$. The non-equivariant splitting provides an $\mathbb{E}_2$-map $\mathrm{MW} \to \mathrm{MU}$ such that the composite $\mathrm{MU} \hookrightarrow \mathrm{MW} \to \mathrm{MU}$ is equivalent to the identity. Using Theorem 6.1.4 we can produce a lift of $\mathrm{MW} \to \mathrm{MU}$ to an $\mathbb{E}_\rho$-map $\mathrm{MW}_{\mathbb{R}} \to \mathrm{MU}_{\mathbb{R}}$. Thus we have an $\mathbb{E}_\rho$-composite $\overline{\mathrm{MU}}_{\mathbb{R}} \to \mathrm{MW}_{\mathbb{R}} \to \mathrm{MU}_{\mathbb{R}}$ whose underlying non-equivariant map is an equivalence. This completes the proof. $\square$

Corollary 7.1.5. Any strongly even $\mathbb{E}_\infty^{C_2}$-ring admits an $\mathbb{E}_\rho$-$\mathrm{MU}_{\mathbb{R}}$-orientation.

Proof. Since $\mathrm{BU}_{\mathbb{R}}$ is grouplike, any $\mathbb{E}_\rho$-map $\mathrm{MU}_{\mathbb{R}} \to E$ is an $\mathbb{E}_\rho$-$\mathrm{MU}_{\mathbb{R}}$-orientation (Lemma 4.2.6). By Theorem 7.1.3 there is a $\mathbb{E}_\infty^{C_2}$-ring map $\mathrm{MW}_{\mathbb{R}} \to E$. Precomposing this with the map from Proposition 7.1.4 gives the result. $\square$

Remark 7.1.6. This can be used to simplify Chadwick–Mandell's proof [CM15, Theorem 1.2] that every complex orientation of an even $\mathbb{E}_2$-ring lifts to an $\mathbb{E}_2$-orientation in the setting of even $\mathbb{E}_\infty$-rings E. Indeed, any such a ring admits an $\mathbb{E}_\infty$-map $\mathrm{MW} \to E$ which we can precompose with $\mathrm{MU} \to \mathrm{MW}$ giving an $\mathbb{E}_2$-orientation of E. So for the comparison map

$$\mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_2}(\mathbf{Sp})}(\mathrm{MU}, E) \to \mathrm{Map}_{\mathbf{Alg}_{\mathbb{E}_0}(\mathbf{Sp})}(\mathrm{MU}, E)$$

we can apply the structured Thom isomorphisms which bypasses an induction argument on the Postnikov tower from Chadwick–Mandell and lets us immediately pass to [CM15, Lemma 5.5, Proposition 5.6].

The orientation results can be extended to larger groups.

Corollary 7.1.7. Let $C_2 \leq G$ be a finite group and $E \in \mathbf{Alg}_{\mathbb{E}_\infty^G}(\underline{\mathbf{Sp}}_G)$. Suppose that $\mathrm{Res}^G_{C_2} E$ is a strongly even. Then, there exists an $\mathrm{Coind}^G_{C_2}\,\mathbb{E}_\rho$-algebra map $\mathrm{MU}^{((G))} = N^G_{C_2}\mathrm{MU}_{\mathbb{R}} \to E$.

Proof. By Corollary 7.1.5, there is an $\mathbb{E}_\rho$-map $\mathrm{MU}_{\mathbb{R}} \to \mathrm{Res}^G_{C_2} E$. We then take the composite

$$N^G_{C_2}\mathrm{MU}_{\mathbb{R}} \longrightarrow N^G_{C_2}\,\mathrm{Res}^G_{C_2}\,E \longrightarrow E$$

where the first map is a $\mathrm{Coind}^G_{C_2}\,\mathbb{E}_\rho$-map (Construction 2.2.5) and the second map is the counit of an adjunction $N^G_{C_2} : \mathbf{Alg}_{\mathbb{E}_\infty^{C_2}}(\underline{\mathbf{Sp}}_{C_2}) \rightleftarrows \mathbf{Alg}_{\mathbb{E}_\infty^G}(\underline{\mathbf{Sp}}_G) : \mathrm{Res}^G_{C_2}$, hence an $\mathbb{E}_\infty^G$-map. $\square$

7.2 Refining Multiplicative Real Orientations

Let E be a strongly even $\mathbb{E}_\infty^{C_2}$-ring. There always exists some $\mathbb{E}_\rho$-map $\mathrm{MU}_{\mathbb{R}} \to E$ by Corollary 7.1.5. The main result of this section provides a way to construct preferred maps.

Theorem 7.2.1. Let E be a strongly even $\mathbb{E}_\infty^{C_2}$-ring spectrum. Any $\mathbb{E}_2$-ring map $\mathrm{MU} \to E^e$ lifts uniquely to an $\mathbb{E}_\rho$-ring map $\mathrm{MU}_{\mathbb{R}} \to E$.

Proof. First, we show that any $\mathbb{E}_2$-ring map $\mathrm{MU} \to E^e$ admits a lift to an $\mathbb{E}_\rho$-ring map $\mathrm{MU}_{\mathbb{R}} \to E$. Recall from Proposition 7.1.4 that we have an $\mathbb{E}_\rho$-retraction $\mathrm{MU}_{\mathbb{R}} \to \mathrm{MW}_{\mathbb{R}} \to \mathrm{MU}_{\mathbb{R}}$ lifting an $\mathbb{E}_2$-retraction $\mathrm{MU} \to \mathrm{MW} \to \mathrm{MU}$. Given an $\mathbb{E}_2$-ring map $\mathrm{MU} \to E^e$, the composite

$$\mathrm{MW} \longrightarrow \mathrm{MU} \longrightarrow E^e$$

is also an $\mathbb{E}_2$-ring map. We can apply Theorem 6.1.4 to lift this to an $\mathbb{E}_\rho$-map $\mathrm{MW}_\mathbb{R} \to E$. Here, Theorem 6.1.4 applies since E is a strongly even $\mathbb{E}_\infty^{C_2}$-ring spectrum admitting an $\mathbb{E}_\rho$-$\mathrm{MW}_\mathbb{R}$-orientation (Theorem 7.1.3), and $\mathrm{B}^\rho\Omega^\infty\Sigma^\rho\mathrm{MU}_\mathbb{R}$ has finite type free homology (Remark 7.1.2). Precomposing this lift with $\mathrm{MU}_\mathbb{R} \to \mathrm{MW}_\mathbb{R}$ gives an $\mathbb{E}_\rho$-map $\mathrm{MU}_\mathbb{R} \to E$. By construction, this is a lift of the $\mathbb{E}_2$-map

$$\mathrm{MU} \longrightarrow \mathrm{MW} \longrightarrow \mathrm{MU} \longrightarrow E.$$

Since $\mathrm{MU} \to \mathrm{MW} \to \mathrm{MU}$ is $\mathrm{id}_{\mathrm{MU}}$, the $\mathbb{E}_\rho$-map $\mathrm{MU}_\mathbb{R} \to E$ we produced is indeed a lift of the given $\mathbb{E}_2$-map $\mathrm{MU} \to E^e$.

It remains to show that such lifts are unique. Let $f\colon \mathrm{MU} \to E^e$ be an $\mathbb{E}_2$-ring map. Let $\overline{f}, \tilde{f}\colon \mathrm{MU}_\mathbb{R} \to E$ be two $\mathbb{E}_\rho$-lifts of f. Since $\overline{f}$ and $\tilde{f}$ both restrict to f, their precompositions with $\mathrm{MW}_\mathbb{R} \to \mathrm{MU}_\mathbb{R}$ both restrict to the precomposition of f with $\mathrm{MW} \to \mathrm{MU}$. By Theorem 6.1.4, any $\mathbb{E}_2$-map $\mathrm{MW} \to E^e$ admits a unique lift to an $\mathbb{E}_\rho$-map $\mathrm{MW}_\mathbb{R} \to E$. By precomposing with $\mathrm{MU}_\mathbb{R} \to \mathrm{MW}_\mathbb{R}$ we learn that $\overline{f}$ and $\tilde{f}$ are equivalent. □

In particular, this provides a Real version of Chadwick–Mandell's result [CM15, Theorem 1.2].

Corollary 7.2.2. Let E be a strongly even $\mathbb{E}_\infty^{C_2}$-ring spectrum.

(i) Any homotopy ring map $\mathrm{MU} \to E^e$ lifts uniquely to an $\mathbb{E}_\rho$-ring map $\mathrm{MU}_\mathbb{R} \to E$.

(ii) Any homotopy ring map $\mathrm{MU}_\mathbb{R} \to E$ lifts to an $\mathbb{E}_\rho$-ring map $\mathrm{MU}_\mathbb{R} \to E$.

Proof.

(i) By Chadwick–Mandell $\mathrm{MU} \to E^e$ lifts uniquely to an $\mathbb{E}_2$-ring map $\mathrm{MU} \to E^e$ [CM15, Theorem 1.2]. Apply Theorem 7.2.1.

(ii) Forget down to a homotopy ring map $\mathrm{MU} \to E^e$ and apply (i). □

We can now leverage this result and immediately obtain structured versions of orientations of interest, for example:

Corollary 7.2.3.

(i) The Hahn–Shi Real orientations [HS20] of Lubin–Tate theories $\mathrm{MU}_\mathbb{R} \to E_n$ admit lifts to $\mathbb{E}_\rho$-maps.

(ii) The Hirzebruch level-n genera [Mei23] $\mathrm{MU}_\mathbb{R} \to \mathrm{tmf}_1(n)$ admit lifts to $\mathbb{E}_\rho$-maps for $n > 1$.

Proof. The $\mathbb{E}_\infty^{C_2}$-structures on E_n and $\mathrm{tmf}_1(n)$ come from the cofree structures [HM17, Theorem 2.4, 2.7]. It's a result from Hahn–Shi that E_n is strongly even [HS20, Theorem 1.9], and a result of Meier that $\mathrm{tmf}_1(n)$ is strongly even [Mei23, Theorem 2.22]. □

Corollary 7.2.4. Let $C_2 \leq G$.

(i) The Hahn–Shi Real orientations $\mathrm{MU}^{((G))} = N_{C_2}^G \mathrm{MU}_\mathbb{R} \to E_n$ refine to $\mathrm{Coind}_H^G \mathbb{E}_\rho$-maps.

(ii) Let $n > 1$ and $G = (\mathbb{Z}/n)^\times$. The Hirzebruch level-$n$ genera induce $\mathrm{Coind}_H^G \mathbb{E}_\rho$-maps $\mathrm{MU}^{((G))} \to \mathrm{tmf}_1(n)$ for $n > 1$.

Proof. These orientations are constructed as in Corollary 7.1.7. □

Previously, the Hahn–Shi Real orientations were only known to be $\mathbb{E}_\sigma$ and the normed versions were only known to be homotopy associative [HS20]. The Hirzebruch level n genera were not known to admit any $\mathbb{E}_V$-structure and were only constructed on the level of homotopy commutative C_2-ring spectra [Mei23]. This is in stark contrast to what is known non-equivariantly: Senger shows that the underlying map $\mathrm{MU} \to \mathrm{tmf}_1(n)$ refines to an $\mathbb{E}_\infty$-map [Sen22, Theorem 1.7]; Burklund–Schlank–Yuan show that the Lubin–Tate theories admit $\mathbb{E}_\infty$-complex orientations [BSY22, Corollary 8.13]. We expect that the Real orientations in Corollary 7.2.3 admit further refinements to $\mathbb{E}_\infty^{C_2}$-maps, and intend to return to this problem in future work.

Remark 7.2.5. Carrick showed in recent work that taking equivariant slice filtrations is lax G-symmetric monoidal [Car25, Definition 6.4]. Thus, taking the slice filtration of the orientations constructed in Corollary 7.1.7 yields multiplicative maps in filtered G-spectra. In particular, Corollary 7.2.4 produces $\mathrm{Coind}_H^G\, \mathbb{E}_\rho$-maps of filtered G-spectra $\mathrm{Slice}(\mathrm{MU}^{((G))}) \to \mathrm{Slice}(E_n)$.

Let p be a prime. In [BHLS23, Construction 5.3], Adams operations are constructed on $\mathrm{MU}_{(p)}$ as $\mathbb{E}_\infty$-maps $\mathrm{MU}_{(p)} \to \mathrm{MU}_{(p)}$. We are grateful to Christian Carrick for pointing out to us that Theorem 7.2.1 can be used to construct $\mathbb{E}_\rho$-lifts of these.

Corollary 7.2.6. The Burklund–Hahn–Levy–Schlank [BHLS23] Adams operations on $\mathrm{MU}_{(2)}$ admit lifts to $\mathbb{E}_\rho$-operations on $\mathrm{MU}_{\mathbb{R}(2)}$.

Proof. The proof of Theorem 7.2.1 works in the 2-local case. So we can produce $\mathbb{E}_\rho$-maps $\overline{\psi}^\ell \colon \mathrm{MU}_{\mathbb{R}(2)} \to \mathrm{MU}_{\mathbb{R}(2)}$ lifting the underlying $\mathbb{E}_2$-map of the Burklund–Hahn–Levy–Schlank Adams operations $\psi^\ell \colon \mathrm{MU}_{(2)} \to \mathrm{MU}_{(2)}$. □

For $C_2 \leq G$ applying $N_{C_2}^G$ also yields an $\mathrm{Coind}_{C_2}^G\, \mathbb{E}_\rho$-structured version.

Remark 7.2.7. The Adams operations constructed in [BHLS23] are $\mathbb{E}_\infty$-operations. It is expected that $\mathrm{MU}_{\mathbb{R}(2)}$ should admit $\mathbb{E}_\infty^{C_2}$-lifts of these Adams operations.

7.3 Multiplication on $\mathrm{BP}_{\mathbb{R}}$

Everything is implicitly (2)-local in this subsection.

We use our $\mathrm{MU}_{\mathbb{R}}$ results to learn about the Real Brown–Peterson spectrum $\mathrm{BP}_{\mathbb{R}}$ which can be constructed as a colimit along an iteration of the Real Quillen idempotent [Ara79, Section 7]. The following provides the first structured version of $\mathrm{BP}_{\mathbb{R}}$ and its norms.

Theorem 7.3.1.

(i) The Real Brown–Peterson spectrum $\mathrm{BP}_{\mathbb{R}}$ admits an $\mathbb{E}_\rho$-algebra structure.

(ii) Let $C_2 \leq G$. Then, $\mathrm{BP}^{((G))} = N_{C_2}^G \mathrm{BP}_{\mathbb{R}}$ admits a $\mathrm{Coind}_{C_2}^G\, \mathbb{E}_\rho$-algebra structure.

Proof.

(i) By Theorem 7.2.1, the Real Quillen idempotent lifts to an $\mathbb{E}_\rho$-algebra map $\mathrm{MU}_{\mathbb{R}} \to \mathrm{MU}_{\mathbb{R}}$. The result follows since filtered colimits of algebras are computed on underlying [NS22, Theorem 5.1.4].

(ii) This follows immediately from (i) and Construction 2.2.5. □

Remark 7.3.2. Since the multiplicative structure on $\mathrm{BP}_{\mathbb{R}}$ is constructed as an $\mathbb{E}_\rho$-retract of $\mathrm{MU}_{\mathbb{R}}$, the $\mathbb{E}_\rho$-Adams operations on $\mathrm{MU}_{\mathbb{R}}$ from Corollary 7.2.6 induce $\mathbb{E}_\rho$-Adams operations on $\mathrm{BP}_{\mathbb{R}}$.

In a similar fashion, we deduce a $\mathrm{BP}_{\mathbb{R}}$-version of Theorem 7.2.1.

Corollary 7.3.3. Let E be a 2-local strongly even $\mathbb{E}_\infty^{C_2}$-ring. Any homotopy ring map $\mathrm{BP} \to E^e$ can be lifted to an $\mathbb{E}_\rho$-algebra map $\mathrm{BP}_\mathbb{R} \to E$.

Proof. We can lift the map $\mathrm{MU} \to \mathrm{BP} \to E^e$ to an $\mathbb{E}_\rho$-algebra map $\mathrm{MU}_\mathbb{R} \to E$ by Theorem 7.2.1. Precomposing with $\mathrm{BP}_\mathbb{R} \to \mathrm{MU}_\mathbb{R} \to E$ yields the desired lift. This is really a lift since the underlying map is

$$\mathrm{BP} \longrightarrow \mathrm{MU} \longrightarrow \mathrm{BP} \longrightarrow E^e$$

where the first two maps compose to $\mathrm{id}_{\mathrm{BP}}$, i.e. this composite is the original map. □

This, for example, gives $\mathbb{E}_\rho$-structured $\mathrm{BP}_\mathbb{R}$-versions of the Hahn–Shi Real orientations of Lubin–Tate theories, and the Hirzebruch level-n genera from Corollary 7.2.3.

8 Factorization Homology of Thom Spectra & $\mathbb{E}_V$-quotients

8.1 Real Topological Hochschild Homology of Thom Spectra

This section will reap some benefits of developing the theory of left modules (Section 3.3) for non-trivial bases.

Conditional on monoidal parameterized straightening Horev–Klang–Zou gave formulas for equivariant factorization homology of Thom spectra over $\mathbb{S}$ [HHK+24, Theorem 7.1.1]. In this section – conditional on the monoidal parametrized straightening – we generalize this to equivariant R-module Thom spectra. In particular, this specializes to give formulas for relative versions of Real Topological Hochschild Homology $\mathrm{THR}(-/R)$ of R-module Thom spectra.

Assumption 8.1.1. Let $\mathcal{O}^\otimes \in \mathbf{Op}_{G,\infty}$ and X be an $\mathcal{O}$-monoidal G-space. There is an $\mathcal{O}$-monoidal equivalence $\underline{\mathbf{PSh}}_G^{\mathcal{O}}(X)^\otimes \simeq (\underline{\mathcal{S}}^G_{/X})^\otimes$.[13]

Remark 8.1.2. Assumption 8.1.1, which is an equivariant version of [Ram22, Corollary 4.9], was stated in [HHK+24, Theorem A.6.1] as folklore. In the Horev–Klang–Zou approach to equivariant Thom spectra, this result is essential to constructing the equivariant Thom spectrum functor as a G-symmetric monoidal functor. In the approach we take in this paper, a microcosmic version (Corollary 3.2.3) is sufficient for most applications. However, generalizing [HHK+24, Theorem 7.1.1] requires this stronger form of straightening. We include this section, conditional on Assumption 8.1.1 for two reasons:

- We hope this encourages a written proof of Assumption 8.1.1 appear in the literature.
- The results of [HHK+24] are used extensively in the literature. For those willing to accept Assumption 8.1.1 without proof, this section provides additional computationally useful results.

Let V be a G-representation and $X \in \mathbf{Sp}_G$. We say that X is V-connective if $\pi_k(X^H) = 0$ for all $H \le G$ and $k < \dim\left(V^H\right)$. We will use this terminology to state the following results.

Theorem 8.1.3 (Conditional on Assumption 8.1.1)**.** Let R be an $\mathbb{E}_\infty^G$-ring and V, W be G-representations. Let $f\colon \Omega^{V+W}X \to \underline{\mathrm{Pic}}_G(R)$ be an $\mathbb{E}_{V+W}$-map. Suppose that X is $(V+W)$-connective. Let M be a G-manifold the same dimension as V. Suppose that $M \times W$ embeds equivariantly into $V \times W$, and that there is an equivariant embedding $D(V) \hookrightarrow M$. Then,

$$\int_{M\times W} \mathrm{Th}_G(f) \simeq \mathrm{Th}_G(f) \otimes \Sigma^\infty_+ \underline{\mathrm{Map}}_*\left(M^+ - D(V), \Omega^W X\right)$$

is an equivalence of R-modules.

[13] The left side is equipped with the Day convolution monoidal structure (Theorem 3.2.2) and the right side is equipped with the slice monoidal structure (Definition 3.1.2).

Proof. By Theorem 3.3.14, $\underline{\mathbf{LMod}}_R^G$ is a distributive G-symmetric monoidal G-∞-category. Conditional on Assumption 8.1.1 the proof of [HHK+24, Theorem 7.1.1] goes through mutatis mundatis replacing $\underline{\mathbf{Sp}}_G$ with $\underline{\mathbf{LMod}}_R^G$. □

The main corollary of interest occurs when M is a representation sphere S^V.

Corollary 8.1.4 (Conditional on Assumption 8.1.1)**.** Let R be an $\mathbb{E}_\infty^G$-ring and V be a representation of G. Let $f\colon \Omega^{V+1}X \to \underline{\mathrm{Pic}}_G(R)$ be an $\mathbb{E}_{V+1}$-map. Suppose that X is $(V+1)$-connective. Then, there is an equivalence

$$\int_{S^V\times\mathbb{R}} \mathrm{Th}_G(f) \simeq \mathrm{Th}_G(f)\otimes \Sigma^\infty_+\Omega X$$

of R-modules.

Specializing further to the C_2-sign sphere S^σ we obtain formulas for relative Real Topological Hochschild Homology since $\mathrm{THR} \simeq \int_{S^\sigma}$, see [Hor19, Remark 7.1.2].

Corollary 8.1.5 (Conditional on Assumption 8.1.1)**.** Let R be an $\mathbb{E}_\infty^{C_2}$-ring and consider an $\mathbb{E}_{\sigma+1}$-map $f\colon \Omega^{\sigma+1}X \to \underline{\mathrm{Pic}}_{C_2}(R)$. Suppose that X is $(\sigma+1)$-connective. Then, there is an equivalence

$$\mathrm{THR}(\mathrm{Th}_{C_2}(f)/R) \simeq \mathrm{Th}_{C_2}(f)\otimes \Sigma^\infty_+\Omega X$$

of R-modules.

8.2 Equivariant Thom Spectra as $\mathbb{E}_V$-quotients

Classically, the Hopkins–Mahowald theorem states that $\mathrm{H}\mathbb{F}_p$ can be constructed as a Thom spectrum over Ω^2S^3 [MNN15, Theorem 4.8], [ACB19, Theorem 5.1]. Equivalently, using the theory of $\mathbb{E}_n$-quotients or versal algebras as Thom spectra [ACB19, Section 4], this describes $\mathrm{H}\mathbb{F}_p$ as a free $\mathbb{E}_2$-ring with a null-homotopy of p.

Equivariant versions of the Hopkins–Mahowald theorem were studied by Behrens–Wilson [BW18], Hahn–Wilson [HW20], Devalapurkar [Dev24], and Levy [Lev22], the latter of which ultimately unifies and extends all known results. For any faithful 2-dimensional representation V of a p-group G, Levy shows that $\mathrm{H}\underline{\mathbb{F}}_p$ can be constructed as the equivariant Thom spectrum of a V-fold loop map $\Omega^V\Sigma^V S^1 \to \mathrm{BGL}_1(\mathbb{S}_{(p)})$.

In this section we record, a fact known to experts, that Antolín-Camarena–Barthel's treatment of $\mathbb{E}_n$-quotients in terms of Thom spectra generalizes to the equivariant setting.

Definition 8.2.1. Let V, W be G-representations and $R \in \mathbf{Alg}_{\mathbb{E}_V\otimes\mathbb{E}_1}(\underline{\mathbf{Sp}}_G)$. Consider an element $\chi \in \pi_W^G(R)$. Write $\tilde{\chi}\colon \Sigma^W R \to R$ for the adjoint map of R-modules. We say that $A \in \mathbf{Alg}_{\mathbb{E}_V}(\underline{\mathbf{LMod}}_R)$ is of characteristic χ if the composite $\eta\circ\tilde{\chi}\colon \Sigma^W R \to R \to A$ is null-homotopic where $\eta\colon R\to A$ is the unit.

Definition 8.2.2 ($\mathbb{E}_V$-quotient)**.** Let V, W be G-representations and $R \in \mathbf{Alg}_{\mathbb{E}_V}(\underline{\mathbf{Sp}}_G)$. Consider an element $\alpha \in \pi_W^G(R)$. The $\mathbb{E}_V$-quotient of R by α, denoted $R /\!\!/_{\mathbb{E}_V} \alpha$, is defined as the following pushout in $\mathbb{E}_V$-algebras

$$\begin{tikzcd} \mathrm{free}^{\mathbb{E}_V}(\Sigma^W R) \arrow[r, "\alpha"] \arrow[d, "0"'] & R \arrow[d] \\ R \arrow[r] & R /\!\!/_{\mathbb{E}_V} \alpha. \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

The first hint that $\mathbb{E}_V$-quotients are related to equivariant Thom spectra is that they have Thom isomorphisms.

Lemma 8.2.3. Let V be a G-representation and $\eta\colon R \to A$ be a map of $\mathbb{E}_V \otimes \mathbb{E}_1$-ring spectra. Let $\chi \in \pi_V^G(R)$. Suppose that A has characteristic χ, then there is an equivalence

$$A \otimes_R \mathrm{free}^{\mathbb{E}_V}(\Sigma^{W+1} R) \simeq A \otimes_R (R \mathbin{/\!\!/}_{\mathbb{E}_V} \chi)$$

of $\mathbb{E}_V$-A-algebras.

Proof. The proof is analogous to [ACB19, Lemma 4.5] □

Definition 8.2.4. Let V, W be G-representations and $R \in \mathbf{Alg}_{\mathbb{E}_V \otimes \mathbb{E}_2}(\underline{\mathbf{Sp}}_G)$. Consider a map $f\colon \Sigma S^W \to \mathrm{BGL}_1(R)$ of pointed G-spaces and let $\tilde{f}\colon \Sigma^W R \to R$ the induced map of R-modules. The characteristic of f is defined to be

$$\chi(f) = \begin{cases} \tilde{f} - 1 & \text{if } W = 0, \\ \tilde{f} & \text{if } W \neq 0. \end{cases}$$

Remark 8.2.5. The discrepancy in the definition of the characteristic of f comes from understanding the equivariant analogue of [ACB19, Proposition 4.9]. Working out the argument, we find that the characteristic should be defined as $\widetilde{f} - a_{-W}$ where

$$[a_{-W}] = \left[S^W \xrightarrow{a_{-W}} S^0 \to \Omega^\infty R\right] \in \pi_W^G(R)$$

is induced by the pre-Euler class $a_{-W} : S^W \to S^0$. On the other hand,

$$\pi_W^G(S^0) \cong [S^W, S^0]_*^G \cong \begin{cases} \{\mathrm{id}_{S^0}, *\} & W = 0, \\ * & W \neq 0 \end{cases}$$

by a connectivity argument.

Lemma 8.2.6. Let V, W be G-representations and $R \in \mathbf{Alg}_{\mathbb{E}_V \otimes \mathbb{E}_2}(\underline{\mathbf{Sp}}_G)$. Consider a map $f\colon S^{W+1} \to \mathrm{BGL}_1(R)$ and let $\overline{f}\colon \Omega^V \Sigma^V S^{W+1} \to \mathrm{BGL}_1(R)$ denote the associated V-fold loop map [Jur25, Theorem 3.15]. Then,

$$\mathrm{Th}_G\left(\overline{f}\colon \Omega^V \Sigma^V S^{W+1} \to \mathrm{BGL}_1(R)\right) \simeq R \mathbin{/\!\!/}_{\mathbb{E}_V} \chi(f)$$

in $\mathbf{Alg}_{\mathbb{E}_V}(\mathbf{LMod}_R)$.

Proof. The proof is analogous to [ACB19, Theorem 4.10]. □

Theorem 8.2.7 ([Lev22, Theorem A])**.** Fix a p-group G. Let V be a faithful 2-dimensional representation of G. Then, there is an equivalence of $\mathbb{E}_V$-algebras between $\mathrm{H}\underline{\mathbb{F}}_p$ and $\mathbb{S} \mathbin{/\!\!/}_{\mathbb{E}_V} p$

Proof. In [Lev22, Theorem A] Levy constructs $\mathrm{H}\underline{\mathbb{F}}_p$ as the equivariant Thom spectrum of a V-fold loop map $\Omega^V \Sigma^V S^1 \to \mathrm{BGL}_1(\mathbb{S}_{(p)})$. Our discussion of $\mathbb{E}_V$-algebras verifies the $\mathbb{S} \mathbin{/\!\!/}_{\mathbb{E}_V} p$ description. □

8.3 Nilpotence for $\mathbb{E}_\sigma$-algebras

Non-equivariantly, the Hopkins–Mahowald Theorem (Section 8.2) is intimately connected to nilpotence phenomena; for this perspective we recommend [MNN15, Hah17, Dev24]. Thus, it is natural to ask if the same is true in the equivariant setting. Obvious analogues of the various famous nilpotence theorems are well-known to fail in the Real equivariant setting; see [BGH20, Remark 3.20], and [Car22, Remark 4.3] for counterexamples.

To demonstrate the utility of $\mathbb{E}_V$-ring structures, we show that an $\mathrm{MU}_{\mathbb{R}}$ detects nilpotence for $\mathbb{E}_\sigma$-rings; and that $\mathrm{H}\underline{\mathbb{Z}}$ detects nilpotence for $\mathbb{E}_\sigma \otimes \mathbb{E}_\infty$-rings.

Remark 8.3.1. These results are surely known to the experts. The authors deduced these from a remark in an expository talk by Hahn [Hah20, t=2760] in which he remarks that an $\mathbb{E}_\infty^{C_2}$-ring is trivial if and only if its underlying non-equivariant ring is trivial. Our only contribution is the easy observation that $\mathbb{E}_\sigma$ is sufficient.

Theorem 8.3.2. *Let* $R \in \mathbf{Alg}_{\mathbb{E}_\sigma}(\underline{\mathbf{Sp}}_{C_2})$.

(i) Then $R \otimes \mathrm{MU}_\mathbb{R} \simeq 0$ if and only if $R \simeq 0$.

(ii) Then $R \otimes K(n)_\mathbb{R} \simeq 0$ for all $n \in \mathbb{N}$ if and only if $R \simeq 0$.

(iii) Suppose further that $R \in \mathbf{Alg}_{\mathbb{E}_\sigma \otimes \mathbb{E}_\infty}(\underline{\mathbf{Sp}}_{C_2})$. Then $R \otimes \mathrm{H}\underline{\mathbb{Z}} \simeq 0$ if and only if $R \simeq 0$.

Proof. In each case, the if direction is clear. We will only show the converse directions. Each converse direction uses the same key ingredient: since R is (at least) an $\mathbb{E}_\sigma$-ring, R is a module over $N_e^{C_2}\mathrm{Res}_e^{C_2} R$ [Hor19, Section 7.1];[14] so to show that $R \simeq 0$ it suffices to show that $\mathrm{Res}_e^{C_2} R \simeq 0$.

(i) Suppose $R \otimes \mathrm{MU}_\mathbb{R} \simeq 0$. Hence, $\mathrm{Res}_e^{C_2} R \otimes \mathrm{MU} \simeq 0$. By Devinatz–Hopkins–Smith Nilpotence [DHS88] it follows that $\mathrm{Res}_e^{C_2} R \simeq 0$. Since R is an $\mathbb{E}_\sigma$-ring, it follows that $R \simeq 0$.

(ii) Suppose $R \otimes K(n)_\mathbb{R} \simeq 0$ for all $n \in \mathbb{N}$. Then, $\mathrm{Res}_e^{C_2} R \otimes K(n) \simeq 0$ for all $n \in \mathbb{N}$. By Devinatz–Hopkins–Smith Nilpotence [DHS88] it follows that $\mathrm{Res}_e^{C_2} R \simeq 0$. Since R is an $\mathbb{E}_\sigma$-ring, it follows that $R \simeq 0$.

(iii) Suppose $R \otimes \mathrm{H}\underline{\mathbb{Z}} \simeq 0$. Therefore, $\mathrm{Res}_e^{C_2} R \otimes \mathrm{H}\mathbb{Z} \simeq 0$. Since $\mathrm{Res}_e^{C_2} R$ is an $\mathbb{E}_\infty$-ring, by May Nilpotence [BMMS86, MNN15] it follows that $\mathrm{Res}_e^{C_2} R \simeq 0$. Since R is in particular an $\mathbb{E}_\sigma$-ring, it follows that $R \simeq 0$. □

Remark 8.3.3. This result works in greater generality than just for $\mathbb{E}_\sigma$-rings by generalizing the module result that we used in the proof, see e.g. [Lev22, Corollary 2.2, Remark 2.4.1]. The exact same proof e.g. also works for $\mathrm{MU}^{((G))} = N_{C_2}^G \mathrm{MU}_\mathbb{R}$ instead of $\mathrm{MU}_\mathbb{R}$ under the assumption that R is a module over $N_e^G \mathrm{Res}_e^G R$.

[14] Let us suggestively write that the action map is induced by the embedding $(\coprod_{C_2} \mathbb{R}^1) \sqcup \mathbb{R}^\sigma \hookrightarrow \mathbb{R}^\sigma$ in $\mathbb{E}_\sigma$, see [Hor19, Section 7.1].

Appendix

Section A includes two further checks in equivariant math.

In Section A.1 we spell out that restrictions and norms of G-symmetric monoidal G-∞-categories are symmetric monoidal (ordinary) functors.

In Section A.2 we explain an $\mathbb{E}_\infty^{C_2}$-structured version of the Real J-homomorphism $J_\mathbb{R}$ based on forthcoming work of Brink–Lenz about equivariant J-homomorphisms.

A More Equivariant Higher Algebra

A.1 Monoidality of Restrictions and Norms

We show that the structure maps of G-symmetric monoidal G-∞-categories are compatible with the levelwise symmetric monoidal structures.

Lemma A.1.1. Let $\mathscr{C}^\otimes \in \mathbf{Mon}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_\infty)$. Then, the restriction and norm functors of $\mathscr{C}^\otimes$ are symmetric monoidal.

Proof. Let $H \leq K \leq G$. Recall first that we obtain symmetric monoidal structures on each level by pulling back:

$$\begin{tikzcd} \mathscr{C}_H^\otimes \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{C}^\otimes \arrow[d] \\ \mathrm{Span}(\mathbb{F}) \arrow[r, "G/H"'] & \mathrm{Span}(\mathbb{F}_G) \end{tikzcd}$$

In other words, it is the image of $\mathscr{C}^\otimes$ under the functor

$$\mathbf{Alg}_{\mathrm{Span}(\mathbb{F}_G)}(\mathbf{Cat}_\infty) \to \mathbf{Alg}_{\mathrm{Span}(\mathbb{F})}(\mathbf{Cat}_\infty) \simeq \mathbf{CMon}(\mathbf{Cat}_\infty)$$

by Proposition 3.2.1. So maps $N_H^K \colon \mathscr{C}_H^\otimes \to \mathscr{C}_K^\otimes$ and $\mathrm{Res}_H^K \colon \mathscr{C}_K^\otimes \to \mathscr{C}_H^\otimes$ are induced by the following natural transformations.

$$\begin{tikzcd} \mathrm{Span}(\mathbb{F}) \arrow[r, "G/H"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & \mathrm{Span}(\mathbb{F}_G) \arrow[d, equal] \\ \mathrm{Span}(\mathbb{F}) \arrow[r, "G/K"'] & \mathrm{Span}(\mathbb{F}_G) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Span}(\mathbb{F}) \arrow[r, "G/K"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & \mathrm{Span}(\mathbb{F}_G) \arrow[d, equal] \\ \mathrm{Span}(\mathbb{F}) \arrow[r, "G/H"'] & \mathrm{Span}(\mathbb{F}_G) \end{tikzcd}$$

These are induced by $G/H \to G/K$ resp. $G/K \leftarrow G/H$ in $\mathrm{Span}(\mathbb{F}_G)$. □

A.2 Real J-Homomorphism

There is a Real J-homomorphism $J_\mathbb{R}$ such that $\mathrm{MU}_\mathbb{R} = \mathrm{Th}_{C_2}(J_\mathbb{R})$. This map also appears in the definition of $\mathrm{MW}_\mathbb{R}$ (Construction 7.1.1). To the best of the authors' knowledge, there is no construction of an $\mathbb{E}_\infty^{C_2}$-Real J-homomorphism in the literature. Here, we will explain how to obtain such a map using forthcoming work of Emma Brink and Tobias Lenz. We are grateful to Emma for suggesting this approach.

Fact A.2.1 (Brink–Lenz, forthcoming)**.** There is an $\mathbb{E}_\infty^{C_2}$-map $J_{C_2} \colon \mathrm{BOP}_{C_2} \to \underline{\mathrm{Pic}}_{C_2}(\underline{\mathbf{Sp}}_{C_2})$ whose C_2-Thom spectrum is MOP_{C_2}.

Their result works in much greater generality but this is all we will need.

So, it suffices to construct an $\mathbb{E}_\infty^{C_2}$-map $\mathrm{BU}_\mathbb{R} \to \mathrm{BOP}_{C_2}$. Postcomposing by J_{C_2} yields $J_\mathbb{R}$.

Construction A.2.2. There exist C_2-orthogonal space models $\widetilde{\mathrm{BUP}}_{\mathbb{R}}$ and of $\widetilde{\mathrm{BOP}}_{C_2}$ by Schwede [Sch14, Example VI.7.1][Sch18, Example 2.4.1], which for an inner product space V are given by

$$\widetilde{\mathrm{BUP}}_{\mathbb{R}}(V) = \mathrm{Gr}^{\mathbb{C}}(V^2_{\mathbb{C}}) \quad \text{and} \quad \widetilde{\mathrm{BOP}}_{C_2}(V) = \mathrm{Gr}(V^2),$$

where $(-)_{\mathbb{C}}$ denotes the complexification functor. All the structure maps are defined essentially in the same way. Moreover, the C_2-action on $\widetilde{\mathrm{BUP}}_{\mathbb{R}}(V)$ is by complex conjugation. We define a map of orthogonal spaces $\widetilde{\mathrm{BUP}}_{\mathbb{R}} \to \widetilde{\mathrm{BOP}}_{C_2}$ given by forgetting the complex structure. This is compatible with complex conjugation and since all structure maps are defined in the essentially same way, this defines a map of ultracommutative orthogonal C_2-spaces. In particular, it induces an $\mathbb{E}_\infty^{C_2}$-map $\mathrm{BUP}_{\mathbb{R}} \to \mathrm{BOP}_{C_2}$ by [LLP25]. Precomposing by $\mathrm{BU}_{\mathbb{R}} \to \mathrm{BUP}_{\mathbb{R}}$ yields a map $\mathrm{BU}_{\mathbb{R}} \to \mathrm{BOP}_{C_2}$.

Definition A.2.3. The Real J-homomorphism is the composition

$$J_{\mathbb{R}} \colon \mathrm{BU}_{\mathbb{R}} \to \mathrm{BOP}_{C_2} \xrightarrow{J_{C_2}} \underline{\mathrm{Pic}}_{C_2}(\underline{\mathbf{Sp}}_{C_2})$$

combining Fact A.2.1 and Construction A.2.2.

Index of Notation

For many of the below notions there are versions enhanced with an underline $\underline{\ }$ or a tensor $(-)^{\otimes}$ which denote a parametrized version or a monoidal version. Let $H \leq K \leq G$ be finite groups.

$\mathbf{Alg}_{\mathscr{P}/\mathscr{Q}}(\mathscr{C})$	the functor ∞-category of maps $\mathscr{P}^{\otimes} \to \mathscr{C}^{\otimes}$ over $\mathscr{Q}^{\otimes}$, see Section 2.1
$\mathbf{Alg}_{\mathscr{O}}(\mathscr{C})$	the ∞-category of $\mathscr{O}$-algebras in $\mathscr{C}$, also given by $\mathbf{Alg}_{\mathscr{O}/\operatorname{Span}(\mathbb{F}_G)}(\mathscr{C})$, see Section 2.1
$\underline{\mathbf{Alg}}^{\mathrm{gp}}_{\mathscr{O}}(\underline{\mathcal{S}}_G)$	the G-∞-category of grouplike $\mathscr{O}$-algebras, see Definition 3.4.1
AlgPatt	the ∞-category of algebraic patterns, see Section 2.1
Ar	the arrow ∞-category given by $\operatorname{Fun}([1], -)$
B^V	the V-th delooping functor, see Theorem 3.4.3
C_2	the cyclic group on two elements
$\underline{\mathscr{C}}_{\underline{d}}$	parameterized fiber of a G-∞-category $\underline{\mathscr{C}}$ over d, see Definition 2.3.3
$\mathbf{Cat}_{\infty}$	the ∞-category of ∞-categories
$\mathbf{Cat}_{G,\infty}$	the ∞-category of G-∞-categories
$\operatorname{coCart}(\mathscr{C})$	the subcategory of $\mathbf{Cat}_{\infty/\mathscr{C}}$ spanned by the coCartesian fibrations and maps preserving coCartesian edges
Coind_H^K	the right adjoint of the restriction functor of G-∞-categories, e.g. see Construction 2.2.5
$(-)^{\mathrm{core}}$	the maximal subgroupoid of an ∞-category
CST	the cohomological slice tower, see Construction 5.3.4
$\mathscr{D}_{/A}$	the slice ∞-catgory in $\mathbf{Cat}_{\infty/\mathscr{O}}$ for $\mathscr{O} \in \mathbf{Cat}_{\infty}$ and $A\colon \mathscr{O} \to \mathscr{D}$, see Definition 3.1.2
$\mathscr{E}^{\mathscr{C}}_{\mathscr{B}}$	the cotensor of $\mathscr{C} \in \mathbf{Cat}_{\infty}$ with $(\mathscr{E} \to \mathscr{B}) \in \mathbf{Cat}_{\infty/\mathscr{B}}$, see Proposition 3.1.1
$\mathbb{E}_V$	the equivariant little disk operad for a G-representation V
$\mathbb{E}_{\infty}^G$	the terminal G-∞-operad
$\mathbb{F}_G$	the ∞-category of finite G-sets
$\underline{\mathbb{F}}_{G,*}$	the G-∞-category of finite pointed G-sets
$\mathbf{Fbrs}(\mathscr{O})$	the ∞-category of fibrous $\mathscr{O}$-patterns for an algebraic pattern $\mathscr{O}$
$\operatorname{free}^{\mathbb{E}_V}$	the free $\mathbb{E}_V$-algebra on a G-spectrum
$\underline{\operatorname{Fun}}^{\mathscr{O}}(\mathscr{C}, \mathscr{D})^{\otimes}$	the functor G-∞-category with Day convolution, see Theorem 3.2.2
GL_1	the right adjoint to the inclusion of grouplike algebras, see Definition 3.4.10

gl_1	a delooping of GL_1, see Remark 3.4.13
gr_s	the s-th associated graded piece of a tower
H^V	the V-graded G-equivariant cohomology groups, given by $\pi_{-V}\underline{\mathrm{map}}_G$
Ind_R^A	the left adjoint to the restriction functor $\underline{\mathbf{LMod}}_A \to \underline{\mathbf{LMod}}_R$, given a map $R \to A$
Ind_H^K	the left adjoint of the restriction functor of various G-∞-categories
$\mathrm{Ind}_{\mathscr{O}}^{\mathscr{Q}}$	the left adjoint of restriction of algebra structures, see Construction 4.1.9
Infl_G	the inflation functor from ∞-operads to G-∞-operads, see Construction 2.2.1
$J_{\mathbb{R}}$	the Real J-homomorphism, see Section A.2
$\mathbf{LMod}_R^G(\mathscr{C})$	the ∞-category of left modules over R, see Construction 3.3.3
M	Thom spectrum functor, see Definition 4.1.1
$\underline{\mathrm{Map}}_{\underline{\mathscr{C}}}(-,-)$	the mapping G-space in a G-∞-category $\underline{\mathscr{C}}$
$\underline{\mathrm{map}}_{\underline{\mathscr{C}}}(-,-)$	the mapping G-spectrum in a G-stable G-∞-category $\underline{\mathscr{C}}$
$\mathbf{Mon}_{\mathscr{O}}(\mathbf{Cat}_\infty)$	the subcategory of $\mathbf{Fbrs}(\mathscr{O}^\otimes)$ spanned by the coCartesian fibrations with morphisms the ones preserving coCartesian edges
$\mathbf{Mon}_{\underline{\mathscr{O}}}^{\mathrm{NS}}(\underline{\mathscr{C}})$	the ∞-category of $\underline{\mathscr{O}}$-monoids in $\underline{\mathscr{C}}$ in the Nardin–Shah formalism
N_H^G	the norm functor for $H \leq G$ from equivariant monoidal structures
$\mathbf{Op}_{G,\infty}$	the ∞-category of G-∞-operads, i.e., $\mathbf{Fbrs}(\mathrm{Span}(\mathbb{F}_G))$
$\mathrm{Or}_A^{\mathscr{P}}(f)$	the space of $\mathscr{P}$-A-orientations, see Definition 4.2.3
$\mathbf{Op}_{G,\infty}^{\mathrm{NS}}$	the ∞-category of G-∞-operads in the Nardin–Shah formalism
$\mathbf{Orb}_G$	the orbit category for a finite group G
$P^{\leq \bullet}$	the (regular) slice tower
P_s^s	the slices of the (regular) slice tower
$\underline{\mathrm{Pic}}_G(R)$	the G-space of invertible objects in $\underline{\mathbf{LMod}}_R^G(\mathscr{C})$, see Construction 3.4.16
$\underline{\mathrm{Pic}}_G(R)_{\downarrow A}$	the comma-G-∞-category of $\underline{\mathrm{Pic}}_G(R)$ over $A \in \underline{\mathbf{LMod}}_R$, see Construction 3.4.16
$\underline{\mathbf{PSh}}_G^{\mathscr{O}}(X)^\otimes$	the presheaf G-∞-category with Day convolution, see Theorem 3.2.2
$\mathcal{S}_G, \mathcal{S}^G$	the ∞-category of G-spaces
$\mathbf{Seg}_{\mathscr{O}}(\mathscr{C})$	the ∞-category of $\mathscr{O}$-Segal objects in $\mathscr{C}$
$\mathbf{Sp}_G, \mathbf{Sp}^G$	the ∞-category of G-spectra
$\mathrm{Span}(\mathscr{C})$	the ∞-category of spans of an ∞-category $\mathscr{C}$ with pullbacks

$\underline{\mathrm{St}}$	parameterized straightening functor, see Proposition 2.3.2
$\mathrm{Th}_G(-)$	the Thom spectrum G-functor, see Definition 4.1.1, Construction 4.1.4
$\underline{\mathrm{Un}}$	parametrized unstraightening functor, see Proposition 2.3.2
γ	the composite $\Omega^\infty\Sigma^2\,\mathrm{MU} \to \Omega^\infty\Sigma^2\,\mathrm{ku} \simeq \mathrm{BU} \to \mathrm{Pic}(\mathbf{Sp})$, see Remark 7.1.2
π_V^G	the V-graded G-homotopy groups, given by $\pi_V^G = [S^V, -]_*^G$
ρ	the regular representation of C_2
ρ_G	the regular representation of G
σ	the sign representation of C_2
$\chi(f)$	the characteristic of f, see Definition 8.2.4
よ	the Yoneda embedding
$R /\!\!/_{\mathbb{E}_V} \alpha$	the $\mathbb{E}_V$-quotient of R by α, see Definition 8.2.2
$\langle n \rangle$	the pointed finite set $\{1, 2, \cdots, n\} \sqcup \{*\} \in \mathbb{F}_*$
$\bigoplus_{o \to o'}$	the induced functor $\mathscr{C}_o^\otimes \to \mathscr{C}_{o'}^\otimes$ over $o \to o'$ of an $\mathscr{O}$-monoidal ∞-category $\mathscr{C}^\otimes \to \mathscr{O}^\otimes$, see Section 2.1
$\underline{\bigoplus}_{o \to o'}$	the indexed tensor product, see Definition 2.3.7

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