

Structured Real Snaith Equivalences

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Abstract

We give a short proof of the Real Snaith equivalences and multiplicative refinements thereof. The key ingredient is control over structured Real orientations, which we manage through Wilson space theory. In particular, this machinery can be applied to recover an $\mathbb{E}_{2\rho}$ -algebra structure on Real Brown–Peterson theory. We apply the Real Snaith theorems to compute $\mathrm{THR}(\mathrm{KU}_{\mathbb{R}})$ and $\mathrm{THR}(\mathrm{MUP}_{\mathbb{R}})$. This requires a norm inverted variant of the Real Snaith theorems, which we prove via the nilpotence theorem.



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Snake Coiled around a Pine Tree before the Rising Sun – Keisai Eisen, 1821.

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1 Introduction

The periodic complex K -theory spectrum KU and the periodic complex bordism spectrum MUP play central roles in stable homotopy theory due to their intimate connection to geometry. Indeed, Atiyah and Hirzebruch introduced KU through complex vector bundles with their direct sums [AH59, AH61], while the homotopy groups of MUP are essentially represented by stably complex manifolds up to complex bordism due to Thom [Tho54]. There is another link to geometry provided by theorems of Snaith [Sna79]:

$$KU \simeq \Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}] \quad \text{and} \quad MUP \simeq \Sigma_+^\infty BU[\beta^{-1}],$$

where β is the Bott class, a generator in π_2 . So KU and MUP are connected to the geometry of line bundles and vector bundles.

Structured refinements of these equivalences suggest a study of the multiplicative structures of KU and MUP through the tensor product of vector bundles. However, while $KU \simeq \Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}]$ is an equivalence of \mathbb{E}_∞ -algebras, Hahn and Yuan surprisingly proved that $MUP \simeq \Sigma_+^\infty BU[\beta^{-1}]$ is not an equivalence of \mathbb{E}_5 -algebras [HY20], so that Snaith's equivalence is not feasible to studying the \mathbb{E}_∞ -algebra structure of MUP . Nonetheless, Hahn–Yuan exhibit an equivalence of \mathbb{E}_2 -algebras, allowing us to study the \mathbb{E}_2 -algebra structure through this viewpoint.

The complex numbers take a distinguished role in this story and it is possible to incorporate the geometry of complex conjugation. By framing the equivalences in terms of C_2 -equivariant homotopy theory, we arrive at Real candidates of the Snaith theorems studying Atiyah's K -theory with Reality $KU_{\mathbb{R}}$ and the Araki–Landweber periodic Real bordism spectrum $MUP_{\mathbb{R}}$. By $\mathbb{C}P_{\mathbb{R}}^\infty$ and $BU_{\mathbb{R}}$ we denote C_2 -spaces refining $\mathbb{C}P^\infty$ and BU via the complex conjugation action. The main theorem of this article consists of the Real Snaith equivalences.

Let ρ be the regular representation for the group C_2 . We denote by \mathbb{E}_ρ the corresponding little disk C_2 -operad and by $\mathbb{E}_\infty^{C_2}$ the terminal C_2 -operad. An $\mathbb{E}_\infty^{C_2}$ -ring spectrum is also called C_2 - \mathbb{E}_∞ -ring spectrum and is informally an \mathbb{E}_∞ -algebra together with all norm multiplications and suitable coherences.

Theorem A (Theorem 3.9). There are equivalences

$$KU_{\mathbb{R}} \simeq \Sigma_+^\infty \mathbb{C}P_{\mathbb{R}}^\infty[\beta_{\mathbb{R}}^{-1}] \quad \text{and} \quad MUP_{\mathbb{R}} \simeq \Sigma_+^\infty BU_{\mathbb{R}}[\beta_{\mathbb{R}}^{-1}]$$

of $\mathbb{E}_\infty^{C_2}$ -algebras resp. of \mathbb{E}_ρ -algebras, where $\beta_{\mathbb{R}}$ is a generator in π_ρ .

Disregarding the multiplicative structures, this theorem was already established in the motivic literature by Gepner–Snaith [GS09] and Annala–Hoyois–Iwasa [AI25, AHI25]. Furthermore, Schwede has a Real-global refinement of the KU -Snaith in forthcoming work. Nonetheless, the \mathbb{E}_ρ -structured equivalence is novel and we demonstrate a new and versatile proof strategy. We will apply the multiplicativity in the computation of $\text{THR}(KU_{\mathbb{R}})$ and $\text{THR}(MUP_{\mathbb{R}})$.

Our new short proof is inspired by technology that we developed in [QZ26a]. We observe that $KU_{\mathbb{R}}$ and $MUP_{\mathbb{R}}$ are examples of strongly even C_2 -spectra in the sense of Hill–Meier [HM17]. These enjoy the property that equivalences are detected on underlying non-equivariant spectra. So after verifying that all of the involved C_2 -spectra are strongly even (Proposition 3.4), it suffices to lift the non-equivariant Snaith equivalences to C_2 -equivariant maps.

Such a program was precisely carried out in our previous work [QZ26a], but we must expand on it to be able to control the periodic object $MUP_{\mathbb{R}}$. We realize this through a periodic version of Wilson space theory and simultaneously use this as an opportunity to improve the structured orientations that we produced in [QZ26a].

Structured orientations through Wilson spaces

The key designer spectra in this story are Thom spectra on Wilson spaces. Consider the Conner–Floyd orientation $\gamma: MU \rightarrow ku$. Then, we may construct the Thom spectra (Construction 2.3)

$$\begin{aligned} \text{MW} &:= \text{Th} \left(\Omega^\infty \Sigma^2 MU \xrightarrow{\Omega^\infty \Sigma^2 \gamma} \Omega^\infty \Sigma^2 ku \simeq BU \xrightarrow{J} \text{Pic}(\text{Sp}) \right) \\ \text{MWP} &:= \text{Th} \left(\Omega^\infty MU \xrightarrow{\Omega^\infty \gamma} \Omega^\infty ku \simeq BUP \xrightarrow{J} \text{Pic}(\text{Sp}) \right), \end{aligned}$$

where the non-periodic version MW was first studied by Hahn–Raksit–Wilson in their work on the even filtration [HRW24]. By showing that every even resp. even periodic \mathbb{E}_∞ -ring spectrum admits an \mathbb{E}_∞ -algebra map from MW resp. MWP via Thom spectrum theory (Theorem 2.4) and splitting off MU resp. MUP in a structured way (Corollary 2.8), we are able to produce structured orientations:

Theorem B (Theorem 2.9). Let E be an \mathbb{E}_∞ -ring spectrum.

- (i) If E is even, then there exists an \mathbb{E}_4 -map $\text{MU} \rightarrow E$.
- (ii) If E is even periodic, then there exists an \mathbb{E}_6 -map $\text{MUP} \rightarrow E$.

In particular, any even periodic \mathbb{E}_∞ -ring admits an \mathbb{E}_6 -MU-orientation.

It is possible to provide such orientations directly with a similar argument without adhering to Wilson space theory. We are going this route since the proof ingredients are necessary to study a C_2 -equivariant variant.

Indeed, after defining Real variants $\text{MW}_\mathbb{R}$ and $\text{MWP}_\mathbb{R}$ we are able to use lifting results from [QZ26a], to obtain an analogous result in the C_2 -equivariant setting. This allows us to apply structured Thom isomorphisms and is the key ingredient to obtain full control on structured Real orientations [QZ26a, Theorem F].

The suitable Real generalization of evenness is Hill–Meier’s notion of strongly even C_2 -spectra [HM17, Definition 3.1]. If a strongly even C_2 -spectrum E moreover satisfies $\Sigma^\rho E \simeq E$, then we call it strongly even periodic (Definition 2.15).

We are able to improve on results in [QZ26a] and obtain:

Theorem C (Theorem 2.19). Let E be an $\mathbb{E}_\infty^{C_2}$ -ring spectrum.

- (i) If E is strongly even, then any \mathbb{E}_{2n} -map $\text{MU} \rightarrow \text{Res}_e^{C_2} E$ lifts uniquely to an $\mathbb{E}_{n\rho}$ -map $\text{MU}_\mathbb{R} \rightarrow E$ for $n \leq 2$.
- (ii) If E is strongly even periodic, then any \mathbb{E}_{2n} -map $\text{MUP} \rightarrow \text{Res}_e^{C_2} E$ lifts uniquely to an $\mathbb{E}_{n\rho}$ -map $\text{MUP}_\mathbb{R} \rightarrow E$ for $n \leq 3$.

The periodic version in (ii) is new and previously we had only established (i) for $n \leq 1$ in [QZ26a]. This result produces structured lifts of many orientations of interest. By formal means we can furthermore produce versions for larger groups $G \geq C_2$. Necessary for such a procedure is the functor $\text{Coind}_{C_2}^G : \text{Op}_{C_2, \infty} \rightarrow \text{Op}_{G, \infty}$ from the ∞ -category of C_2 - ∞ -operads to the ∞ -category of G - ∞ -operads, which is the right adjoint of $\text{Res}_{C_2}^G : \text{Op}_{G, \infty} \rightarrow \text{Op}_{C_2, \infty}$.

We write $\text{MU}^{(G)} := N_{C_2}^G \text{MU}_\mathbb{R}$, $\text{MUP}^{(G)} := N_{C_2}^G \text{MUP}_\mathbb{R}$ and $\text{BP}^{(G)} := N_{C_2}^G \text{BP}_\mathbb{R}$. Let us state a number of examples in the general version for larger groups.

Theorem D (Theorem 2.22, Corollary 2.24). Let $G \geq C_2$ be a finite group. Let $n \geq 1$.

- (i) Suppose that $G \leq \mathbb{G}_n$ is a finite subgroup of the extended Morava stabilizer group. There exist $\text{Coind}_{C_2}^G \mathbb{E}_{3\rho}$ -algebra maps $\text{MUP}^{(G)} \rightarrow E_n$ to Lubin–Tate theory.
- (ii) Let $G = (\mathbb{Z}/n)^\times$. Then, the Hirzebruch level- n -genera $\text{MU}_\mathbb{R} \rightarrow \text{tmf}_1(n)$ induce $\text{Coind}_{C_2}^G \mathbb{E}_{2\rho}$ -algebra maps $\text{MU}^{(G)} \rightarrow \text{tmf}_1(n)$.
- (iii) The G -spectrum $\text{BP}^{(G)}$ admits a $\text{Coind}_{C_2}^G \mathbb{E}_{2\rho}$ -algebra structure.

In particular, the Real Brown–Peterson spectrum $\text{BP}_\mathbb{R}$ admits an $\mathbb{E}_{2\rho}$ -algebra structure.

In [QZ26a] we already obtained versions of these orientations with a $\text{Coind}_{C_2}^G \mathbb{E}_\rho$ -structure. There, we relied on Chadwick–Mandell’s result saying that every complex orientation lifts uniquely to an \mathbb{E}_2 -complex orientation [CM15]. This allowed us to immediately apply the lifting result. On the other hand, not every complex orientation lifts to an \mathbb{E}_4 -complex orientation, e.g. it is still not known whether the Quillen idempotent $\text{MU} \rightarrow \text{MU}$ lifts to an \mathbb{E}_4 -algebra map. As such, we need to input strong non-equivariant theorems first to obtain specific \mathbb{E}_4 -lifts before lifting them to the equivariant setting via Theorem C.

For example, the $\mathbb{E}_{2\rho}$ -algebra structure on $\text{BP}_\mathbb{R}$ is obtained by lifting the \mathbb{E}_4 -idempotent on $\text{MU}_{(2)}$ provided by Basterra–Mandell [BM13]. Here, we want to clarify that the $\mathbb{E}_{2\rho}$ -algebra structure on $\text{BP}_\mathbb{R}$ is also established in concurrent work by Carrick–Hill–Stewart and the authors, which will appear in forthcoming work (Remark 2.23). Therefore, we provide another proof of this result.

Real Snaith equivalences & Real topological Hochschild homology

By writing down a lift of the \mathbb{E}_∞ -Snaith equivalence $KU \simeq \Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}]$ and by lifting the \mathbb{E}_2 -Snaith equivalence $MUP \rightarrow \Sigma_+^\infty BU[\beta^{-1}]$ from Hahn–Yuan, we thus prove the structured Real Snaith equivalences with the outline suggested after [Theorem A](#). So we prove

$$KU_{\mathbb{R}} \simeq \Sigma_+^\infty \mathbb{C}P_{\mathbb{R}}^\infty[\beta_{\mathbb{R}}^{-1}] \quad \text{and} \quad MUP_{\mathbb{R}} \simeq \Sigma_+^\infty BU_{\mathbb{R}}[\beta_{\mathbb{R}}^{-1}]$$

of $\mathbb{E}_\infty^{C_2}$ -algebras resp. of \mathbb{E}_ρ -algebras.

Our main application of C_2 -equivariant multiplicative structures lies in Real trace theory [[HM26](#), [DMPR21](#)]. A C_2 -equivariant refinement of topological Hochschild homology is Real topological Hochschild homology defined by

$$\mathrm{THR}(R) := R \otimes_{N_e^{C_2} \mathrm{Res}_e^{C_2} R} R$$

for C_2 -ring spectra R with $N_e^{C_2} \mathrm{Res}_e^{C_2} R$ - $N_e^{C_2} \mathrm{Res}_e^{C_2} R$ -bimodule structures, which is e.g. provided by an \mathbb{E}_σ -algebra structure. Here, σ is the C_2 -sign representation. This is the entry point in approaching Real algebraic K -theory, a C_2 -equivariant refinement of algebraic K -theory that furthermore encodes the information of Hermitian K -theory [[HM26](#), [CDH⁺23](#), [CDH⁺25](#), [CDH⁺26](#), [AKKQ25](#)].

A central computation in the theory of THH is Bökstedt’s theorem, i.e. the computation of $\mathrm{THH}(\mathbb{F}_p)$, which can be stated as $\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Sigma_+^\infty \Omega S^3$. As such, a similar level of importance should be given to its Real analog $\mathrm{THR}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Sigma_+^\infty \Omega S^{1+\rho}$, which was proven by Dotto–Moi–Patchkoria–Reeh [[DMPR21](#)]. Using the same strategy as for the Real Snaith theorems, we prove a more structured version of the Real Bökstedt theorem by lifting an \mathbb{E}_2 -equivalence by Bayındır–Moulinos [[BM22](#)].

Theorem E ([Theorem 4.1](#)). There is an equivalence $\mathrm{THR}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Sigma_+^\infty \Omega S^{1+\rho}$ of \mathbb{E}_ρ -algebras.

This demonstrates that our techniques in proving the Real Snaith theorems are versatile enough to be applicable in various frameworks.

Our main application of the Real Snaith equivalences lies in the computation of $\mathrm{THR}(KU_{\mathbb{R}})$ and $\mathrm{THR}(MUP_{\mathbb{R}})$.

Theorem F ([Theorem 4.7](#)). There are equivalences

$$\mathrm{THR}(KU_{\mathbb{R}}) \simeq KU_{\mathbb{R}} \otimes \Sigma_+^\infty B^\rho U_{\mathbb{R}}(1) \quad \text{resp.} \quad \mathrm{THR}(MUP_{\mathbb{R}}) \simeq MUP_{\mathbb{R}} \otimes \Sigma_+^\infty B^\rho U_{\mathbb{R}}$$

of $\mathbb{E}_\infty^{C_2}$ -algebras resp. of \mathbb{E}_1 -algebras.

Non-equivariantly, such a computation using Snaith’s theorem was first carried out by Stonek [[Sto20](#), [RSV22](#)], who used that THH commutes with inverting elements and that the THH of spherical group rings was well-known. The THR of spherical group rings is also established – however, more care is needed in commuting the localization past THR.

Theorem G ([Proposition 4.3](#)). Let R be an $\mathbb{E}_\infty^{C_2}$ -ring spectrum and V be a C_2 -representation. Suppose that $\bar{x} \in \pi_V^{C_2}(R)$ and $x = \mathrm{Res}_e^{C_2} \bar{x} \in \pi_{|V|}^e(R)$. Then, there is an equivalence

$$\mathrm{THR}\left(R[(\mathrm{Nm}_e^{C_2} x)^{-1}]\right) \simeq \mathrm{THR}(R)[(\mathrm{Nm}_e^{C_2} x)^{-1}]$$

of $\mathbb{E}_\infty^{C_2}$ -algebras.

In particular, the Real Snaith equivalences do not naively lend themselves to a THR computation since the Real Bott element $\beta_{\mathbb{R}}$ is not of the form required in [Theorem G](#). Our solution is to improve on the Real Snaith theorems by providing the following norm inverted variants.

Theorem H ([Theorem 3.19](#)). There are equivalences

$$KU_{\mathbb{R}} \simeq \Sigma_+^\infty \mathbb{C}P_{\mathbb{R}}^\infty[\mathrm{Nm}_e^{C_2}(\beta)^{-1}] \quad \text{and} \quad MUP_{\mathbb{R}} \simeq \Sigma_+^\infty BU_{\mathbb{R}}[\mathrm{Nm}_e^{C_2}(\beta)^{-1}]$$

of $\mathbb{E}_\infty^{C_2}$ - resp. \mathbb{E}_ρ -algebras.

In proving this theorem, we do not explicitly compute $\mathrm{Nm}_e^{C_2}(\beta)$, but rather proceed indirectly via an application of the nilpotence theorem on C_2 -geometric fixed points. Another proof ingredient is the Real Snaith theorem itself.

Combining these equivalences with the THR computation of spherical group rings allows us to carry out the computation in [Theorem F](#).

Outline

The remainder of the article is organized as follows:

In [Section 2](#) we use Wilson space theory to produce \mathbb{E}_4 - resp. \mathbb{E}_6 -orientations of even resp. even periodic \mathbb{E}_∞ -ring spectra. We lift these to a C_2 -equivariant setting and prove a lifting theorem for structured orientations. The rest of the article can be read independently of this section by accepting [Theorem 2.19](#).

In [Section 3](#) we prove the structured Real Snaith theorems and their norm-inverted variants. We spell out the real Snaith theorems as consequences.

In [Section 4](#) we analyze when one can commute THR with inverting an element. We then use this to compute $\mathrm{THR}(\mathrm{KU}_{\mathbb{R}})$ and $\mathrm{THR}(\mathrm{MUP}_{\mathbb{R}})$. We also prove an \mathbb{E}_ρ -version of the Real Bökstedt theorem.

Notations & Conventions

We collect some notational conventions here that we will use throughout the whole article.

- (1) Throughout, G will always be a finite group.
- (2) We denote by ρ and σ the regular and sign representations of C_2 respectively.
- (3) Let $X, Y \in \mathrm{Sp}_G$. We denote by $\underline{\mathrm{map}}_G(X, Y)$ the mapping G -spectrum refining the mapping G -space $\underline{\mathrm{Map}}_G(X, Y)$. We also write $\underline{\mathrm{map}}_{\mathrm{Sp}_G}(X, Y)$ and $\underline{\mathrm{Map}}_{\mathrm{Sp}_G}(X, Y)$.
- (4) The G - ∞ -operad \mathbb{E}_∞^G is the terminal G - ∞ -operad. So \mathbb{E}_∞^G -algebras admit all norms – they are also known as normed algebras, ultracommutative algebras, or G - \mathbb{E}_∞ -algebras.
- (5) There is a restriction functor of equivariant ∞ -operads $\mathrm{Res}_H^G: \mathrm{Op}_{G, \infty} \rightarrow \mathrm{Op}_{H, \infty}$ which admits a right adjoint $\mathrm{Coind}_H^G: \mathrm{Op}_{H, \infty} \rightarrow \mathrm{Op}_{G, \infty}$, the coinduction functor. See [\[Ste25, Section 1.3.1\]](#) and [\[QZ26a, Remark 2.2.5\]](#) for a discussion in the language of parametrized higher categories.

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2 Structured Real orientations through Wilson spaces

2.1 Recollection on lifting equivariant maps

We recall our lifting theorems from [\[QZ26a\]](#), since they will be applied in the text and motivate our discussion of Wilson space theory. Let us denote by ρ the regular representation of the group C_2 . Our first general lifting result is the following.

Theorem 2.1 ([\[QZ26a, Theorem D\]](#)). Let $E \in \mathrm{Sp}^{C_2}$ be strongly even and $X \in \mathrm{Sp}^{C_2}$. Suppose the following:

- (i) The C_2 -spectrum X is slice bounded below.
- (ii) There is an indexing set I such that $\mathrm{HZ} \otimes X \simeq \mathrm{HZ} \otimes \bigoplus_{i \in I} S^{n_i \rho}$ for certain $n_i \in \mathbb{N}$ where every degree appears finitely often.

Then, every map $\mathrm{Res}_e^{C_2} X \rightarrow \mathrm{Res}_e^{C_2} E$ lifts uniquely to a C_2 -equivariant map $X \rightarrow E$.

Using this general lifting result, we can deduce a structured lifting result. Concerning equivariant multiplications, let us recall that every C_2 -representation V gives rise to an equivariant little disk operad \mathbb{E}_V . We denote by $\mathbb{E}_\infty^{C_2}$ the terminal C_2 -operad, its algebras are also called C_2 - \mathbb{E}_∞ -algebras in the literature. Informally, those are \mathbb{E}_∞ -algebras together with all norm multiplications and possible coherences. Furthermore, we will write Th_{C_2} for the C_2 -equivariant Thom spectrum functor, for example constructed through parametrized colimits [\[HHK⁺24, QZ26a\]](#).

Theorem 2.2 ([QZ26a, Theorem E]). Let R be an $\mathbb{E}_\infty^{C_2}$ -ring spectrum and E be a strongly even $\mathbb{E}_\infty^{C_2}$ -algebra in $\underline{\mathbf{LMod}}_R$. Suppose that X is an $n\rho$ -loop space with a map $f: X \rightarrow \underline{\mathbf{Pic}}_{C_2}(R)$ of $n\rho$ -loop spaces. Suppose the following:

- (i) There exists an $\mathbb{E}_{n\rho}$ - R -algebra map $\mathrm{Th}_{C_2}(f) \rightarrow E$.
- (ii) There is an indexing set I such that $\mathbb{H}\mathbb{Z} \otimes B^{n\rho}X \simeq \mathbb{H}\mathbb{Z} \otimes \bigoplus_{i \in I} S^{n_i\rho}$ for certain $n_i \in \mathbb{N}$ where every degree appears finitely often.

Then, every \mathbb{E}_{2n} -map $\mathrm{Res}_e^{C_2} \mathrm{Th}_{C_2}(f) \rightarrow \mathrm{Res}_e^{C_2} E$ lifts uniquely to an $\mathbb{E}_{n\rho}$ -map $\mathrm{Th}_{C_2} f \rightarrow E$.

The main crux to lifting from Thom spectra consists of the existence of a structured orientation (Theorem 2.2(i)), since it allows us to perform structured Thom isomorphisms. So structured Real orientations $\mathrm{MU}_\mathbb{R} \rightarrow E$ depend crucially on the existence of a structured map $\mathrm{MU}_\mathbb{R} \rightarrow E$. The theory of Wilson spaces is what allows us to access such orientations.

In [QZ26a] we laid out such a theory for \mathbb{E}_ρ -orientations. The goal of the current section is to expand on this theory to $\mathbb{E}_{2\rho}$ -orientations and to give a periodic variant of it. This will be a theory of $\mathbb{E}_{3\rho}$ - $\mathrm{MUP}_\mathbb{R}$ -orientations and it will be relevant to access the Real Snaith theorem.

2.2 Structured orientations through Wilson spaces

Inspired by [HRW24, Theorem 3.2.10] we will provide new structured complex orientations of even (periodic) \mathbb{E}_∞ -ring spectra. The key will be to manipulate the remarkable obstruction-theoretic features of the Wilson spaces $W_{2i} := \Omega^\infty \Sigma^{2i} \mathrm{MU}$ that were first studied by Steve Wilson in his PhD thesis [Wil73].

Construction 2.3. Let $\gamma: \mathrm{MU} \rightarrow \mathrm{ku}$ be a map that is surjective on homotopy.³ We consider the Thom spectra

$$\begin{aligned} \mathbf{MW} &:= \mathrm{Th} \left(\Omega^\infty \Sigma^2 \mathrm{MU} \xrightarrow{\Omega^\infty \Sigma^2 \gamma} \Omega^\infty \Sigma^2 \mathrm{ku} \simeq \mathrm{BU} \xrightarrow{J} \mathrm{Pic}(\mathrm{Sp}) \right), \\ \mathbf{MWP} &:= \mathrm{Th} \left(\Omega^\infty \mathrm{MU} \xrightarrow{\Omega^\infty \gamma} \Omega^\infty \mathrm{ku} \simeq \mathrm{BUP} \xrightarrow{J} \mathrm{Pic}(\mathrm{Sp}) \right). \end{aligned}$$

The spectrum \mathbf{MW} was introduced by Hahn–Raksit–Wilson in [HRW24, Definition 3.2.9], while \mathbf{MWP} is a natural periodic version of it.

Recall that an even \mathbb{E}_∞ -ring spectrum E is called *even periodic* if $\Sigma^2 E \simeq E$ of E -modules.

Theorem 2.4. Let E be an \mathbb{E}_∞ -ring spectrum.

- (i) If E is even, then there exists an \mathbb{E}_∞ -map $\mathbf{MW} \rightarrow E$.
- (ii) If E is even periodic, then there exists an \mathbb{E}_∞ -map $\mathbf{MWP} \rightarrow E$.

Proof. For expository purposes, we will recite [HRW24, Proof of Theorem 3.2.10] and [AKKQ25, Theorem 5.10] in (i), even though the proof technique of (ii) also applies for (i). The same technique will not be feasible to conclude (ii).

- (i) The spectrum MU is connective and has an even cell decomposition,⁴ so Thomifying $\Omega^\infty \Sigma^2(-)$ of that cell decomposition yields an even \mathbb{E}_∞ -cell decomposition of \mathbf{MW} with a single 0-cell. The obstructions to the extensions

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & E \\ \downarrow & \nearrow & \\ \mathbf{MW} & & \end{array}$$

as \mathbb{E}_∞ -maps lie in odd homotopy groups of E , which vanish.

- (ii) Since \mathbf{MWP} is unbounded, it seems hard to mimic cell attaching arguments here. We will employ a Thom spectrum argument instead.

An \mathbb{E}_∞ -ring map $\mathbf{MWP} \rightarrow E$ is the same datum as a nullhomotopy of

³Such a map exists, and is e.g. realized by the Conner–Floyd orientation.

⁴The even cell decomposition comes from the computation $H_* \mathrm{MU} \cong \mathbb{Z}[b_1, b_2, \dots]$. This also reveals why the same argument does not work for ku and $\Omega^\infty \Sigma^2 \mathrm{ku} \simeq \mathrm{BU}$, as $H_* \mathrm{ku}$ is considerably more complicated.

$$\mathrm{MU} \xrightarrow{\gamma} \mathrm{ku} \xrightarrow{J} \mathrm{pic} \mathbb{S} \longrightarrow \mathrm{pic} E,$$

see e.g. [ACB19, Lemma 3.15]. So we need to show that this composite is nullhomotopic. Consider the fiber sequence

$$\Sigma \mathrm{gl}_1 E \simeq \tau_{\geq 1}(\mathrm{pic} E) \longrightarrow \mathrm{pic} E \longrightarrow \pi_0(\mathrm{pic} E)$$

coming from the standard t -structure of Sp . We wish to show that our aforementioned composite factors through $\Sigma \mathrm{gl}_1 E \rightarrow \mathrm{pic} E$. The composite

$$\mathrm{ku} \xrightarrow{J} \mathrm{pic} \mathbb{S} \longrightarrow \mathrm{pic} E \longrightarrow \pi_0(\mathrm{pic} E)$$

factors through $\mathrm{ku} \rightarrow \tau_{\leq 0} \mathrm{ku} \simeq \pi_0 \mathrm{ku}$ since $\pi_0(\mathrm{pic} E)$ is 0-truncated. The map $\pi_0 \mathrm{ku} \rightarrow \pi_0(\mathrm{pic} E)$ is described by sending a complex vector space V to $\Sigma^{|V|} E$. On the other hand, $\Sigma^{|V|} E \simeq E$ by even periodicity of E . So the map $\pi_0 \mathrm{ku} \rightarrow \pi_0(\mathrm{pic} E)$ is the null map.

Thus, there exists a factorization

$$\begin{array}{ccccc} & & \exists & \xrightarrow{\quad} & \Sigma \mathrm{gl}_1 E \\ & & \curvearrowright & & \downarrow \\ \mathrm{MU} & \xrightarrow{\gamma} & \mathrm{ku} & \xrightarrow{J} & \mathrm{pic} \mathbb{S} \longrightarrow \mathrm{pic} E \end{array}$$

An Atiyah–Hirzebruch spectral sequence computation shows that $\mathrm{map}_{\mathrm{Sp}}(\mathrm{MU}, \Sigma \mathrm{gl}_1 E)$ is odd, see [QZ26a, Proposition B.2.3]. So every map $\mathrm{MU} \rightarrow \Sigma \mathrm{gl}_1 E$ is nullhomotopic. \square

As we have indicated, Theorem 2.4(i) was known due to Hahn–Raksit–Wilson [HRW24]. However, the \mathbb{E}_∞ -MWP-orientation required a new argument.

Our next goal is to provide structured maps $\mathrm{MU} \rightarrow \mathrm{MW}$ and $\mathrm{MUP} \rightarrow \mathrm{MWP}$. To get started on this, we will need a quick lemma.

Lemma 2.5. Consider two maps $f: x \rightarrow z$ and $g: y \rightarrow z$ of even connective spectra. Suppose that $H^*(x; A)$ is even for every finitely generated abelian group A , that $\pi_* y$ and $\pi_* z$ are of finite type and that $\pi_* g$ is surjective. Then, there exists a lift

$$\begin{array}{ccc} & & y \\ & \nearrow & \downarrow g \\ x & \xrightarrow{f} & z \end{array}$$

of connective spectra.

Proof. The cofiber $\mathrm{cofib} g$ in $\mathrm{Sp}_{\geq 0}$ is formed in Sp . There is an exact sequence

$$\pi_0 \mathrm{map}_{\mathrm{Sp}}(x, y) \longrightarrow \pi_0 \mathrm{map}_{\mathrm{Sp}}(x, z) \longrightarrow \pi_0 \mathrm{map}_{\mathrm{Sp}}(x, \mathrm{cofib} g),$$

so it suffices to show $\pi_0 \mathrm{map}_{\mathrm{Sp}}(x, \mathrm{cofib} g) = 0$. On the other hand, $\mathrm{cofib} g$ is odd by evenness of y, z and surjectivity of $\pi_* g$. An Atiyah–Hirzebruch spectral sequence style argument e.g. as in [QZ26a, Proposition B.2.3] allows us to conclude. \square

Proposition 2.6. There are equivalences

$$\mathrm{MW} \simeq \mathrm{MU} \otimes \Sigma_+^\infty \mathrm{fib}(\Omega^\infty \Sigma^2 \gamma) \quad \text{resp.} \quad \mathrm{MWP} \simeq \mathrm{MUP} \otimes \Sigma_+^\infty \mathrm{fib}(\Omega^\infty \gamma)$$

of \mathbb{E}_4 - resp. \mathbb{E}_6 -ring spectra.

Proof. Let's consider MW . It suffices to construct an \mathbb{E}_4 -section of $\Omega^\infty \Sigma^2 \gamma: \Omega^\infty \Sigma^2 \mathrm{MU} \rightarrow \mathrm{BU}$. To construct this, we will provide a section of the map

$$\Omega^\infty \Sigma^6 \gamma: \Omega^\infty \Sigma^6 \mathrm{MU} \longrightarrow \mathrm{B}^4 \mathrm{BU} \simeq \mathrm{BU}\langle 6 \rangle$$

of pointed spaces and take Ω^4 afterwards. Consider the diagram

$$\begin{array}{ccc}
 & \Omega^\infty \Sigma^6 \text{MU} & \\
 & \nearrow & \downarrow \Omega^\infty \Sigma^6 \gamma \\
 \text{BU}\langle 6 \rangle & \xlongequal{\quad} & \text{BU}\langle 6 \rangle
 \end{array}$$

We obtain such a lift by Lemma 2.5. Here, the choice of γ such that $\pi_*\gamma$ is surjective crucially enters, and moreover, the even cohomology of $\text{BU}\langle 6 \rangle$ was computed by Stong [Sto63] and Singer [Sin68], see also [AHS01, Section 4.5].

For MWP we need to provide an \mathbb{E}_6 -section of $\Omega^\infty \gamma: \Omega^\infty \text{MU} \rightarrow \text{BUP}$. But we have provided a section of

$$\Omega^\infty \Sigma^6 \gamma: \Omega^\infty \Sigma^6 \text{MU} \longrightarrow \text{B}^4 \text{BU} \simeq \text{BU}\langle 6 \rangle$$

above and can take Ω^6 of this. \square

Remark 2.7. A splitting $\text{MW} \simeq \text{MU} \otimes_{\Sigma_+^\infty} \text{fib}(\Omega^\infty \Sigma^2 \gamma)$ of \mathbb{E}_2 -ring spectra was already provided by [HRW24, Theorem 3.2.10] using the exact same argument. Our only observation is that it can be delooped twice more. Nonetheless, we decided to carefully repeat the proof to make sure that the refinement is valid. In particular, one has to be more careful since $\text{B}^4 \text{BU} \simeq \text{BU}\langle 6 \rangle$ is more delicate than $\text{B}^2 \text{BU} \simeq \text{BSU}$.

Corollary 2.8. There is an \mathbb{E}_4 - resp. \mathbb{E}_6 -map

$$\text{MW} \longrightarrow \text{MW} \quad \text{resp.} \quad \text{MWP} \longrightarrow \text{MWP}$$

which are idempotents underlying and split off MU resp. MUP as an \mathbb{E}_4 - resp. \mathbb{E}_6 -retract.

Proof. We apply the previous proposition (Proposition 2.6) and the fact that algebras in \mathcal{S} are (multiplicatively) augmented, so we construct the idempotents as

$$\begin{aligned}
 \text{MW} &\simeq \text{MU} \otimes_{\Sigma_+^\infty} \text{fib}(\Omega^\infty \Sigma^2 \gamma) \rightarrow \text{MU} \otimes_{\Sigma_+^\infty} * \rightarrow \text{MU} \otimes_{\Sigma_+^\infty} \text{fib}(\Omega^\infty \Sigma^2 \gamma) \simeq \text{MW} \\
 \text{MWP} &\simeq \text{MUP} \otimes_{\Sigma_+^\infty} \text{fib}(\Omega^\infty \gamma) \rightarrow \text{MUP} \otimes_{\Sigma_+^\infty} * \rightarrow \text{MUP} \otimes_{\Sigma_+^\infty} \text{fib}(\Omega^\infty \gamma) \simeq \text{MWP}
 \end{aligned}$$

by augmentation followed by the unit. \square

Combining the two main observations of this section (Theorem 2.4, Corollary 2.8) we are able to produce structured orientations.

Theorem 2.9. Let E be an \mathbb{E}_∞ -ring spectrum.

- (i) If E is even, then there exists an \mathbb{E}_4 -map $\text{MU} \rightarrow E$.
- (ii) If E is even periodic, then there exists an \mathbb{E}_6 -map $\text{MUP} \rightarrow E$.

In particular, any even periodic \mathbb{E}_∞ -ring admits an \mathbb{E}_6 -MU-orientation.

Proof. We construct them as compositions

$$\text{MU} \longrightarrow \text{MW} \longrightarrow E \quad \text{resp.} \quad \text{MUP} \longrightarrow \text{MWP} \longrightarrow E$$

of maps from Theorem 2.4 and Corollary 2.8. Since the preferred map $\text{MU} \rightarrow \text{MUP}$ is an \mathbb{E}_∞ -ring map, we also deduce the last sentence of the assertion. \square

One could instead also have run an argument as in Theorem 2.4 by considering 6-fold deloopings to construct these orientations directly. Nonetheless, we will mimic Theorem 2.4 to prove a C_2 -equivariant analog and Corollary 2.8 will be necessary in the proof of the C_2 -equivariant analog (Theorem 2.19). That's why we are taking the route through Wilson spaces.

As an immediate example we obtain a structured orientation on the Lubin–Tate theories constructed by the Goerss–Hopkins–Miller theorem [Rez98, GH04, Lur18].

Corollary 2.10. Let k be a perfect field and Γ_h be a height h formal group law over k . Let $E(k, \Gamma_h)$ be the associated Lubin–Tate theory. There exists an \mathbb{E}_6 -ring map $\text{MUP} \rightarrow E(k, \Gamma_h)$.

Remark 2.11. In the case that k is algebraically closed, Burklund–Schlank–Yuan have already shown that there exists an \mathbb{E}_∞ -ring map $\text{MUP} \rightarrow E(k, \Gamma_h)$ through the chromatic Nullstellensatz [BSY22, Corollary 8.13].

Remark 2.12. This article is about Snaith’s equivalence $\text{MUP} \simeq \Sigma_+^\infty \text{BU}[\beta^{-1}] =: \text{MUP}_{\text{Snaith}}$ for a certain Bott class $\beta \in \pi_2(\Sigma_+^\infty \text{BU})$, see [Sna79]. As mentioned in Section 1, Hahn–Yuan discovered that this equivalence surprisingly does not refine to an equivalence of \mathbb{E}_∞ -ring spectra. In fact, they show that there does not exist an \mathbb{E}_5 -equivalence $\text{MUP} \simeq \text{MUP}_{\text{Snaith}}$, and they provide an \mathbb{E}_2 -equivalence [HY20].

Hahn–Yuan then ask whether there exists an \mathbb{E}_∞ -algebra map $\text{MU} \rightarrow \text{MUP}_{\text{Snaith}}$. Our results are thus a first step towards answering this question, namely we provide an \mathbb{E}_6 -algebra map, in fact there even exists an \mathbb{E}_6 -algebra map $\text{MUP} \rightarrow \text{MUP}_{\text{Snaith}}$ by Theorem 2.9.

Our structured orientations came from structured decompositions of MW and MWP, so it is natural to ask how much these can be refined.

Question 2.13. We have constructed \mathbb{E}_4 - and \mathbb{E}_6 -decompositions

$$\text{MW} \simeq \text{MU} \otimes_{\Sigma_+^\infty} \text{fib}(\Omega^\infty \Sigma^2 \gamma) \quad \& \quad \text{MWP} \simeq \text{MUP} \otimes_{\Sigma_+^\infty} \text{fib}(\Omega^\infty \gamma).$$

Do they admit even more structure?

Since further deloopings of $\text{BU}\langle 6 \rangle$ do not have cohomology concentrated in even degrees [Sto63], the obstruction-theoretic arguments as written in Proposition 2.6 do not work anymore.

2.3 Structured Real orientations

In [QZ26a] we developed an obstruction theory for lifting non-equivariant structured orientations to structured equivariant orientations. One of our main ingredients to handle the Thom spectrum $\text{MU}_\mathbb{R}$ was a C_2 -equivariant analog of the Wilson space result Theorem 2.9. At the time, we were following Hahn–Raksit–Wilson and were under the impression that MW only splits off MU as an \mathbb{E}_2 -retract (Remark 2.7). With the realization that an \mathbb{E}_4 -retract is possible as well as the discussion of a periodic version (Theorem 2.9), we are thus able to improve upon our results. As an application we will for example be able to produce an $\mathbb{E}_{2\rho}$ -algebra structure on $\text{BP}_\mathbb{R}$, see Theorem 2.22.

The Real analogs of MW and MWP from Construction 2.3 are the following C_2 -Thom spectra.

Construction 2.14. Let $\gamma_\mathbb{R}: \text{MU}_\mathbb{R} \rightarrow \text{ku}_\mathbb{R}$ be a map such that $\pi_{*\rho} \gamma_\mathbb{R}$ surjective.⁵ We consider the C_2 -Thom spectra

$$\begin{aligned} \text{MW}_\mathbb{R} &:= \text{Th}_{C_2} \left(\Omega^\infty \Sigma^\rho \text{MU}_\mathbb{R} \xrightarrow{\Omega^\infty \Sigma^\rho \gamma_\mathbb{R}} \Omega^\infty \Sigma^\rho \text{ku}_\mathbb{R} \simeq \text{BU}_\mathbb{R} \xrightarrow{J_\mathbb{R}} \underline{\text{Pic}}(\underline{\text{Sp}}^{C_2}) \right), \\ \text{MWP}_\mathbb{R} &:= \text{Th}_{C_2} \left(\Omega^\infty \text{MU}_\mathbb{R} \xrightarrow{\Omega^\infty \gamma_\mathbb{R}} \Omega^\infty \text{ku}_\mathbb{R} \simeq \text{BUP}_\mathbb{R} \xrightarrow{J_\mathbb{R}} \underline{\text{Pic}}(\underline{\text{Sp}}^{C_2}) \right). \end{aligned}$$

The C_2 -spectrum $\text{MW}_\mathbb{R}$ was considered by Angelini–Knoll–Kong–Quigley [AKKQ25, Construction 5.8] and $\text{MWP}_\mathbb{R}$ is a natural periodic analog.

Definition 2.15. Let $R \in \text{Alg}_{\mathbb{E}_\infty}(\text{Sp}^{C_2})$. It is said to be **strongly even periodic** if it is strongly even and $\Sigma^\rho X \simeq X$ of R -modules.

In particular, the underlying spectrum of a strongly even periodic C_2 -spectrum is an even periodic spectrum.

Theorem 2.16. Let E be an $\mathbb{E}_\infty^{C_2}$ -ring spectrum.

- (i) If E is strongly even, then there exists an $\mathbb{E}_\infty^{C_2}$ -map $\text{MW}_\mathbb{R} \rightarrow E$.
- (ii) If E is strongly even periodic, then there exists an $\mathbb{E}_\infty^{C_2}$ -map $\text{MWP}_\mathbb{R} \rightarrow E$.

Proof. Part (i) already appeared in [AKKQ25, Theorem 5.10]. Let us only discuss (ii), whose proof strategy also applies to (i). We essentially mimic our proof in Theorem 2.4.

An $\mathbb{E}_\infty^{C_2}$ -ring map $\text{MWP}_\mathbb{R} \rightarrow E$ is the same datum as a nullhomotopy of

⁵Such a map exists, e.g. by lifting the Conner–Floyd orientation, see e.g. [QZ26a, Corollary 6.2.2].

$$\mathrm{MU}_{\mathbb{R}} \xrightarrow{\gamma_{\mathbb{R}}} \mathrm{ku}_{\mathbb{R}} \xrightarrow{J_{\mathbb{R}}} \underline{\mathrm{pic}}(\mathbb{S}) \longrightarrow \underline{\mathrm{pic}}(E)$$

by [QZ26a, Lemma 4.2.7], where $\underline{\mathrm{pic}}$ denotes the corresponding connective C_2 -spectrum. So we need to show that this composite is nullhomotopic. There is a fiber sequence

$$\Sigma \mathrm{gl}_1 E \simeq \tau_{\geq 1}(\underline{\mathrm{pic}}(E)) \longrightarrow \underline{\mathrm{pic}}(E) \longrightarrow \pi_0(\underline{\mathrm{pic}}(E))$$

coming from the standard t -structure of Sp^{C_2} . The first equivalence is by $\tau_{\geq 1}\underline{\mathrm{pic}}(E) \simeq \Sigma \mathrm{gl}_1 E$ using the natural equivalence $\Omega \mathrm{pic}(\mathcal{C}) \simeq \mathrm{aut}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$ for any symmetric monoidal ∞ -category \mathcal{C} , see e.g. [MS16, Section 2.2]. We now wish to show that our aforementioned composite factors through $\Sigma \mathrm{gl}_1 E \rightarrow \underline{\mathrm{pic}}(E)$. The composite

$$\mathrm{ku}_{\mathbb{R}} \xrightarrow{J_{\mathbb{R}}} \underline{\mathrm{pic}}(\mathbb{S}) \longrightarrow \underline{\mathrm{pic}}(E) \longrightarrow \pi_0(\underline{\mathrm{pic}}(E))$$

factors through $\mathrm{ku}_{\mathbb{R}} \rightarrow \tau_{\leq 0} \mathrm{ku}_{\mathbb{R}} \simeq \pi_0 \mathrm{ku}_{\mathbb{R}}$ since $\pi_0(\underline{\mathrm{pic}}(E))$ is 0-truncated. The map $\pi_0 \mathrm{ku}_{\mathbb{R}} \rightarrow \pi_0(\underline{\mathrm{pic}}(E))$ is now nullhomotopic, as we may check levelwise:

- On π_0^e we obtain the J -homomorphism $\pi_0 \mathrm{ku} \rightarrow \pi_0(\mathrm{pic} E^e)$, $1 \mapsto \mathbb{S}^2 \otimes E$, which is null by even periodicity of E^e .
- On $\pi_0^{C_2}$ we obtain the map $\pi_0 \mathrm{ko} \rightarrow \pi_0(\mathrm{pic} E)$, $1 \mapsto \mathbb{S}^{\rho} \otimes E$ which is null by strong even periodicity of E .

Let us justify that this map sends 1 to $\mathbb{S}^{\rho} \otimes E$. It factors through $\pi_0(\mathrm{pic} \mathbb{S}_{C_2}) \rightarrow \pi_0(\mathrm{pic} E)$, so it suffices to show that $\pi_0 \mathrm{ko} \rightarrow \pi_0(\mathrm{pic} \mathbb{S}_{C_2})$ sends 1 to \mathbb{S}^{ρ} . Since the Real J -homomorphism is an $\mathbb{E}_{\infty}^{C_2}$ -algebra map [QZ26a, Appendix A.2], the diagram

$$\begin{array}{ccc} \pi_0 \mathrm{ko} & \longrightarrow & \pi_0(\mathrm{pic} \mathbb{S}_{C_2}) \\ \mathrm{Nm}_e^{C_2} \uparrow & & \uparrow \mathrm{Nm}_e^{C_2} \\ \pi_0 \mathrm{ku} & \longrightarrow & \pi_0(\mathrm{pic} \mathbb{S}) \end{array}$$

commutes. The norm map $\mathrm{Nm}_e^{C_2} : \pi_0 \mathrm{ku} \rightarrow \pi_0 \mathrm{ko}$ sends 1 to 2. Going the other way, the composite is given by $1 \mapsto \mathbb{S}^2 \mapsto \mathbb{S}^{2\rho}$. We deduce $2 \mapsto \mathbb{S}^{2\rho}$. On the other hand, $\pi_0(\mathrm{pic} \mathbb{S}_{C_2}) \cong \mathbb{Z} \times \mathbb{Z}$ is known to only consist of the $\mathrm{RO}(C_2)$ -graded spheres by [tD79, Theorem 5] and [FLM01, Theorem 0.1]. Thus, we must have $1 \mapsto \mathbb{S}^{\rho}$.

Thus, there exists a factorization

$$\begin{array}{ccccc} & & \exists & \dashrightarrow & \Sigma \mathrm{gl}_1 E \\ & & & & \downarrow \\ \mathrm{MU}_{\mathbb{R}} & \xrightarrow{\gamma_{\mathbb{R}}} & \mathrm{ku}_{\mathbb{R}} & \xrightarrow{J_{\mathbb{R}}} & \underline{\mathrm{pic}}(\mathbb{S}) \longrightarrow \underline{\mathrm{pic}}(E) \end{array}$$

By [QZ26a, Theorem 5.3.8] the shift of the C_2 -mapping spectrum $\Omega \underline{\mathrm{map}}_{\mathrm{Sp}^{C_2}}(\mathrm{MU}_{\mathbb{R}}, \Sigma \mathrm{gl}_1 E)$ is strongly even. By Greenlees' gap characterization [Gre18, Lemma 1.2] we conclude

$$\pi_0^{C_2} \underline{\mathrm{map}}_{\mathrm{Sp}^{C_2}}(\mathrm{MU}_{\mathbb{R}}, \Sigma \mathrm{gl}_1 E) \simeq \pi_{-1}^{C_2} \Omega \underline{\mathrm{map}}_{\mathrm{Sp}^{C_2}}(\mathrm{MU}_{\mathbb{R}}, \Sigma \mathrm{gl}_1 E) \simeq 0.$$

Hence, any map $\mathrm{MU}_{\mathbb{R}} \rightarrow \Sigma \mathrm{gl}_1 E$ is nullhomotopic. \square

Similar to the non-equivariant case (Theorem 2.4), let us stress again that the $\mathbb{E}_{\infty}^{C_2}$ - $\mathrm{MW}_{\mathbb{R}}$ -orientation is already known due to Angelini-Knoll-Kong-Quigley [AKKQ25, Theorem 5.10], while the $\mathbb{E}_{\infty}^{C_2}$ - $\mathrm{MWP}_{\mathbb{R}}$ -orientation is really a new result that required a number of equivariant theorems, such as those developed in [QZ26a].

Proposition 2.17. There are equivalences

$$\mathrm{MW}_{\mathbb{R}} \simeq \mathrm{MU}_{\mathbb{R}} \otimes \Sigma_+^{\infty} \mathrm{fib}(\Omega^{\infty} \Sigma^{\rho} \gamma_{\mathbb{R}}) \quad \text{resp.} \quad \mathrm{MWP}_{\mathbb{R}} \simeq \mathrm{MUP}_{\mathbb{R}} \otimes \Sigma_+^{\infty} \mathrm{fib}(\Omega^{\infty} \gamma_{\mathbb{R}})$$

of $\mathbb{E}_{2\rho}$ - resp. $\mathbb{E}_{3\rho}$ -ring spectra.

Proof. Provided the $\mathbb{E}_\infty^{C_2}$ -orientations from [Theorem 2.16](#) we can now lift the idempotents from [Corollary 2.8](#) with the exact same argument as in [[QZ26a](#), Proposition 6.1.4]. \square

Corollary 2.18. There is an $\mathbb{E}_{2\rho}$ - resp. $\mathbb{E}_{3\rho}$ -map

$$\mathrm{MW}_{\mathbb{R}} \longrightarrow \mathrm{MW}_{\mathbb{R}} \quad \text{resp.} \quad \mathrm{MWP}_{\mathbb{R}} \longrightarrow \mathrm{MWP}_{\mathbb{R}}$$

which are idempotents underlying and split off $\mathrm{MU}_{\mathbb{R}}$ resp. $\mathrm{MUP}_{\mathbb{R}}$ as an $\mathbb{E}_{2\rho}$ - resp. $\mathbb{E}_{3\rho}$ -retract.

Proof. We apply the previous proposition ([Proposition 2.17](#)) and the fact that algebras in \mathcal{S}_{C_2} are (multiplicatively) augmented, so we construct the idempotents as

$$\begin{aligned} \mathrm{MW}_{\mathbb{R}} &\simeq \mathrm{MU}_{\mathbb{R}} \otimes \Sigma_+^\infty \mathrm{fib}(\Omega^\infty \Sigma^\rho \gamma_{\mathbb{R}}) \rightarrow \mathrm{MU}_{\mathbb{R}} \otimes \Sigma_+^\infty * \rightarrow \mathrm{MU}_{\mathbb{R}} \otimes \Sigma_+^\infty \mathrm{fib}(\Omega^\infty \Sigma^\rho \gamma_{\mathbb{R}}) \simeq \mathrm{MW}_{\mathbb{R}} \\ \mathrm{MWP}_{\mathbb{R}} &\simeq \mathrm{MUP}_{\mathbb{R}} \otimes \Sigma_+^\infty \mathrm{fib}(\Omega^\infty \gamma_{\mathbb{R}}) \rightarrow \mathrm{MUP}_{\mathbb{R}} \otimes \Sigma_+^\infty * \rightarrow \mathrm{MUP}_{\mathbb{R}} \otimes \Sigma_+^\infty \mathrm{fib}(\Omega^\infty \gamma_{\mathbb{R}}) \simeq \mathrm{MWP}_{\mathbb{R}} \end{aligned}$$

by augmentation followed by the unit. \square

Theorem 2.19. Let E be an $\mathbb{E}_\infty^{C_2}$ -ring spectrum.

- (i) If E is strongly even, then any \mathbb{E}_{2n} -map $\mathrm{MU} \rightarrow \mathrm{Res}_e^{C_2} E$ lifts uniquely to an $\mathbb{E}_{n\rho}$ -map $\mathrm{MU}_{\mathbb{R}} \rightarrow E$ for $n \leq 2$.
- (ii) If E is strongly even periodic, then any \mathbb{E}_{2n} -map $\mathrm{MUP} \rightarrow \mathrm{Res}_e^{C_2} E$ lifts uniquely to an $\mathbb{E}_{n\rho}$ -map $\mathrm{MUP}_{\mathbb{R}} \rightarrow E$ for $n \leq 3$.

Proof. Given the $\mathbb{E}_\infty^{C_2}$ -orientations from [Theorem 2.16](#) and the $\mathbb{E}_{2\rho}$ - and $\mathbb{E}_{3\rho}$ -splittings from [Corollary 2.18](#) we can run the same argument as in [[QZ26a](#), Theorem 6.2.1] \square

Remark 2.20. In particular, any strongly even $\mathbb{E}_\infty^{C_2}$ -ring spectrum admits an $\mathbb{E}_{2\rho}$ - $\mathrm{MU}_{\mathbb{R}}$ -orientation and any strongly even periodic $\mathbb{E}_\infty^{C_2}$ -ring spectrum admits an $\mathbb{E}_{3\rho}$ - $\mathrm{MU}_{\mathbb{R}}$ -orientation using [Theorem 2.9](#).

This allows us to improve all the consequences that we obtained in [[QZ26a](#)]. However, we need to exercise care: The lifting in [[QZ26a](#)] was immediate to do: Let E be a strongly even $\mathbb{E}_\infty^{C_2}$ -ring spectrum. Using a result from Chadwick–Mandell [[CM15](#), Theorem 1.2] we may lift any complex orientation of $\mathrm{Res}_e^{C_2} E$ to an \mathbb{E}_2 -orientation of $\mathrm{Res}_e^{C_2} E$, which we may then lift to an \mathbb{E}_ρ -orientation of E by [Remark 2.20](#), see [[QZ26a](#), Corollary 6.2.2]. However, Chadwick–Mandell’s result does not generalize to \mathbb{E}_4 -orientations [[CM15](#), Theorem 1.4], so one needs to input additional (typically hard) theorems to obtain an \mathbb{E}_4 -orientation on the underlying.

Example 2.21. In [[QZ26a](#), Corollary 6.2.3, Corollary 6.2.6] we obtained \mathbb{E}_ρ -versions of all of the following. We can lift the \mathbb{E}_4 - and the \mathbb{E}_6 -orientations from [Theorem 2.9](#), but there are also other explicit orientations from the literature that we can lift.

- (i) Let k be a perfect field of characteristic 2 and Γ_h be a formal group law of height h over k . Hahn–Shi constructed Real orientations of Lubin–Tate theories $\mathrm{MU}_{\mathbb{R}} \rightarrow E(k, \Gamma_h)$ in [[HS20](#), Theorem 1.1], connecting the obstruction-theoretic C_2 -action on $E(k, \Gamma_h)$ with the geometry of complex conjugation. In particular, they showed that $E(k, \Gamma_h)$ is strongly even [[HS20](#), Theorem 1.9].

In their work on the telescope conjecture, Burklund–Hahn–Levy–Schlank constructed certain \mathbb{E}_4 -orientations on $E(\overline{\mathbb{F}}_p, \Gamma_h)$, see [[BHLS23](#), Corollary 5.12]. Moreover, in his work on elliptic genera, Senger also obtained \mathbb{E}_∞ - MU -orientations of height ≤ 2 Lubin–Tate theories $E(k, \Gamma_h)$ for $k \subseteq \overline{\mathbb{F}}_p$ a field of characteristic p which is algebraic over \mathbb{F}_p , see [[Sen22](#), Corollary 1.5].

Moving to MUP -orientations, Balderrama has constructed \mathbb{E}_∞ - MUP -orientations for height ≤ 2 Lubin–Tate theories [[Bal23](#), Theorem 6.5.3]. At long last, via the chromatic Nullstellensatz [[BSY22](#), Corollary 8.13], one may obtain an \mathbb{E}_∞ -orientation $\mathrm{MUP} \rightarrow E(k, \Gamma_h)$ for every height h when k is algebraically closed.

By [Theorem 2.19](#) we may lift these to $\mathbb{E}_{2\rho}$ -orientations $\mathrm{MU}_{\mathbb{R}} \rightarrow E(k, \Gamma_h)$ resp. to $\mathbb{E}_{3\rho}$ -orientations $\mathrm{MUP}_{\mathbb{R}} \rightarrow E(k, \Gamma_h)$.

- (ii) Let $n \geq 0$. Meier refined the Hirzebruch level- n -genera to C_2 -equivariant maps $\mathrm{MU}_{\mathbb{R}} \rightarrow \mathrm{tmf}_1(n)$ and particularly proved that $\mathrm{tmf}_1(n)$ is strongly even [Mei23, Theorem 2.22]. Senger lifted the Hirzebruch level- n genus to an \mathbb{E}_{∞} -map $\mathrm{MU} \rightarrow \mathrm{tmf}_1(n)$, so we can lift this to an $\mathbb{E}_{2\rho}$ -map $\mathrm{MU}_{\mathbb{R}} \rightarrow \mathrm{tmf}_1(n)$ by Theorem 2.19.

This is a first approximation to Senger’s question [Sen22, Question 1.10], about whether there is a C_2 -equivariant analog for his \mathbb{E}_{∞} -orientations. We will return to fully answering Senger’s question in the future.

- (iii) Burklund–Hahn–Levy–Schlank used the Adams conjecture to consider certain \mathbb{E}_{∞} -algebra maps $\Psi^{\ell}: \mathrm{MU}_{(2)} \rightarrow \mathrm{MU}_{(2)}$, which they call *Adams operations* on $\mathrm{MU}_{(2)}$, see [BHLS23, Construction 5.3]. So Theorem 2.19 lifts these to $\mathbb{E}_{2\rho}$ -maps $\mathrm{MU}_{\mathbb{R}(2)} \rightarrow \mathrm{MU}_{\mathbb{R}(2)}$.⁶

Theorem 2.22. The Real Brown–Peterson spectrum $\mathrm{BP}_{\mathbb{R}}$ admits an $\mathbb{E}_{2\rho}$ -algebra structure.

Proof. Basterra–Mandell constructed an \mathbb{E}_4 -idempotent $\mathrm{MU} \rightarrow \mathrm{MU}$ which splits off a copy of BP , see [BM13, Theorem 1.1]. By Theorem 2.19 we may lift this to an $\mathbb{E}_{2\rho}$ -algebra map $e: \mathrm{MU}_{\mathbb{R}} \rightarrow \mathrm{MU}_{\mathbb{R}}$. We form the filtered colimit

$$\mathrm{BP}_{\mathbb{R}}^{\mathrm{BS}} := \mathrm{colim} \left(\mathrm{MU}_{\mathbb{R}} \xrightarrow{e} \mathrm{MU}_{\mathbb{R}} \xrightarrow{e} \cdots \right) \in \mathrm{Alg}_{\mathbb{E}_{2\rho}}(\mathrm{Sp}^{C_2})$$

in $\mathbb{E}_{2\rho}$ -algebras. There are now several ways to see that its underlying C_2 -spectrum is $\mathrm{BP}_{\mathbb{R}}$.

As a filtered colimit of strongly even C_2 -spectra, $\mathrm{BP}_{\mathbb{R}}^{\mathrm{BS}}$ is strongly even again. Moreover, its underlying spectrum is BP , which is Landweber exact. Thus, it is Real Landweber exact by [HM17, Theorem 3.6(b)]. There is only one strongly even C_2 -spectrum which is Real Landweber exact and underlying BP by [HM17, Proposition 3.13] and $\mathrm{BP}_{\mathbb{R}}$ satisfies these conditions. Thus, $\mathrm{BP}_{\mathbb{R}}^{\mathrm{BS}}$ is an $\mathbb{E}_{2\rho}$ -refinement of $\mathrm{BP}_{\mathbb{R}}$. \square

By formal means [QZ26a, Construction 2.2.5, Corollary 6.1.6] we obtain higher group versions of the previous results.

Remark 2.23. In [QZ26a] we showed that $\mathrm{BP}_{\mathbb{R}}$ admits an \mathbb{E}_{ρ} -algebra structure. Moreover, the techniques there showed that it suffices to compute the \mathbb{Z} -valued Bredon cohomology of $\mathrm{BU}_{\mathbb{R}}\langle 3\rho \rangle := \Omega^{\infty} \Sigma^{3\rho} \mathrm{ku}_{\mathbb{R}}$, see [QZ26a, Theorem 5.4.3]. This computation was carried out by Carrick–Hill–Stewart and the authors and will appear in forthcoming work. So Theorem 2.22 is an independent second proof of this result.

Corollary 2.24. Let $G \geq C_2$ be a finite group.

- (i) Let $G \leq \mathbb{G}_h$ be a finite subgroup of the extended Morava stabilizer group. There exist $\mathrm{Coind}_{C_2}^G \mathbb{E}_{3\rho}$ -algebra maps $\mathrm{MUP}^{(G)} \rightarrow E(k, \Gamma_h)$, where Γ_h is a formal group law of height h over an algebraically closed field k .
- (ii) Let $n \geq 0$ and $G = (\mathbb{Z}/n)^{\times}$. Then, the Hirzebruch level- n -genera induce $\mathrm{Coind}_{C_2}^G \mathbb{E}_{2\rho}$ -algebra maps $\mathrm{MU}^{(G)} \rightarrow \mathrm{tmf}_1(n)$.
- (iii) The G -spectrum $\mathrm{BP}^{(G)}$ admits a $\mathrm{Coind}_{C_2}^G \mathbb{E}_{2\rho}$ -algebra structure.

Such orientations of Lubin–Tate theory are of great interest to be able to understand the action of the Morava stabilizer group. A further task is to orient it from smaller objects such as $\mathrm{BP}^{(G)}\langle m \rangle$. We carry this out in forthcoming work of ours [QZ26b] building on work of Beaudry–Hill–Shi–Zeng [BHSZ21].

3 Real Snaith equivalences

3.1 Real Snaith theorems

Snaith proved that the important chromatic spectra KU and MUP can be recovered as the localizations

$$\mathrm{KU} \simeq \Sigma_+^{\infty} \mathrm{BU}(1)[\beta^{-1}] \quad \text{and} \quad \mathrm{MUP} \simeq \Sigma_+^{\infty} \mathrm{BU}[\beta^{-1}]$$

for a certain Bott class β . While the first equivalence can be refined to one of \mathbb{E}_{∞} -ring spectra, the second one turned out not to enjoy a similar property. Hahn–Yuan proved that it does not refine to an equivalence of \mathbb{E}_5 -ring spectra, and they at least provide an \mathbb{E}_2 -equivalence [HY20]. Using Wilson space

⁶Note here that our results also work for (2)-localized versions.

machinery we already produced a structured map $\text{MUP} \rightarrow \Sigma_+^\infty \text{BU}[\beta^{-1}]$, more precisely an \mathbb{E}_6 -map, see Remark 2.12.

Motivated by this observation and our lifting results [QZ26a], we will lift Snaith's equivalences to C_2 -equivariant homotopy theory.

Construction 3.1.

(i) The inclusion in the second summand

$$\varphi: \Sigma^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty \longrightarrow \mathbb{S} \oplus \Sigma^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty \simeq \Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty.$$

restricts along the bottom cell inclusion to a map $\Sigma^\infty S^\rho \simeq \Sigma^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^1 \rightarrow \Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty$. This yields the **Real Bott element** $\beta_\mathbb{R} \in \pi_\rho(\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty)$.

(ii) Postcomposing by the preferred map $\Sigma_+^\infty \text{BU}_\mathbb{R}(1) \rightarrow \Sigma_+^\infty \text{BU}_\mathbb{R}$ yields $\beta_\mathbb{R} \in \pi_\rho(\Sigma_+^\infty \text{BU}_\mathbb{R})$.

Remark 3.2. Let X be a pointed G -space. We should note that the map $\Sigma^\infty X \rightarrow \Sigma_+^\infty X$ is not induced by the inclusion map $X \rightarrow X_+$ since this is not a map of pointed G -spaces. Instead, consider the split fiber sequence

$$\mathbb{S} \xrightarrow{i} \Sigma_+^\infty X \xrightarrow{p} \Sigma^\infty X.$$

Let $r: \Sigma_+^\infty X \rightarrow \mathbb{S}$ denote the right-inverse of i . Then, there is a factorization

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{i} & \Sigma_+^\infty X & \longrightarrow & \Sigma^\infty X \\ & & \searrow & & \downarrow \exists! \\ & & \text{id}_{\Sigma_+^\infty X} - i \circ r & & \Sigma_+^\infty X \end{array}$$

by the universal property of cofibers. The map φ from Construction 3.1 is an example of this.

By construction, the Real Bott elements $\beta_\mathbb{R}$ restrict to the classical non-equivariant Bott elements $\beta \in \pi_2(\Sigma_+^\infty \mathbb{C}\mathbb{P}^\infty)$ and $\beta \in \pi_2(\Sigma_+^\infty \text{BU})$. Inverting those classes yields Real candidates for Real Snaith equivalences: $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}]$ and $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$. These are \mathbb{E}_∞ -algebras in C_2 -spectra, and we will later be able to show that they admit $\mathbb{E}_\infty^{C_2}$ -algebra structures (Proposition 3.8).

Lemma 3.3. The C_2 -spectra $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}]$ and $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$ are Real orientable.

Proof. Let us only discuss $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$, the same argument works for $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}]$. For this, we observe that the composite

$$\Sigma^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty \xrightarrow{\varphi} \Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty \longrightarrow \Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}] \xrightarrow{\beta_\mathbb{R}^{-1}} \Sigma^\rho \Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}],$$

is a Real orientation of $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$ by construction. Indeed, precomposing by $\Sigma^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^1$ yields the following diagram:

$$\begin{array}{ccccccc} \Sigma^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^1 & \longrightarrow & \Sigma^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty & \xrightarrow{\varphi} & \Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty & \longrightarrow & \Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}] \xrightarrow{\beta_\mathbb{R}^{-1}} \Sigma^\rho \Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}], \\ & & & & \searrow & & \uparrow \\ & & & & \beta_\mathbb{R} & & \end{array}$$

By construction, the first three maps compose to $\beta_\mathbb{R}$, so the composition is $\beta_\mathbb{R}^{-1} \beta_\mathbb{R} = 1$. \square

This sets us up to prove the following stronger result. The proof strategy is inspired by [CHY24, Theorem 3.3].

Proposition 3.4. The C_2 -spectra $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}]$ and $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$ are strongly even periodic.

Proof. Let us demonstrate a proof for $\Sigma_+^\infty \mathrm{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$, it goes through analogously for $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}]$. It suffices to prove that these are strongly even, the periodicity comes from the fact that $\beta_\mathbb{R}$ is an invertible class in degree ρ .

Real orientability (Lemma 3.3) yields a map $f: \mathrm{MU}_\mathbb{R} \rightarrow \Sigma_+^\infty \mathrm{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$ of homotopy commutative C_2 -ring spectra [HK01]. Then, the composite

$$\begin{array}{ccc} \mathbb{S} \otimes \Sigma_+^\infty \mathrm{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}] & \xrightarrow{1 \otimes \mathrm{id}} & \mathrm{MU}_\mathbb{R} \otimes \Sigma_+^\infty \mathrm{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}] & \xrightarrow{f \otimes \mathrm{id}} & \Sigma_+^\infty \mathrm{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}] \otimes \Sigma_+^\infty \mathrm{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}] \\ & & & & \downarrow \text{mult} \\ & & & & \Sigma_+^\infty \mathrm{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}] \end{array}$$

exhibits $\Sigma_+^\infty \mathrm{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$ as a retract of $\mathrm{MU}_\mathbb{R} \otimes \Sigma_+^\infty \mathrm{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$. On the other hand, $\mathrm{MU}_\mathbb{R} \otimes \Sigma_+^\infty \mathrm{BU}_\mathbb{R}$ is strongly even by Real orientation theory [Hil22, Theorem 4.28].

Greenlees' gap characterization [Gre18, Lemma 1.2] describes strongly even C_2 -spectra as those C_2 -spectra X with $\pi_{*\rho-i}^{C_2} X = 0$ for $i = 1, 2, 3$. Since the Real Bott element is in degree ρ , the gap condition remains after localizing at the Real Bott element. \square

In the presence of equivariant multiplicative structures, norms will show up. On the level of homotopy groups two different (but related) constructions can be meant by this, so we explain our notation to be able to take particular care about this.

Notation 3.5. Let R be an $\mathbb{E}_\infty^{C_2}$ -ring and $n \in \mathbb{Z}$. Suppose that $y \in \pi_n^e(R)$.

- (i) We write $N_e^{C_2} y \in \pi_{n\rho}^{C_2}(N_e^{C_2} R)$ for the element that arises after applying the functor $N_e^{C_2}$ to the map $y: \mathbb{S}^n \rightarrow R$.
- (ii) We write $\mathrm{Nm}_e^{C_2} y \in \pi_{n\rho}^{C_2}(R)$ for the element that arises after applying the norm multiplication $N_e^{C_2} \mathrm{Res}_e^{C_2} R \rightarrow R$ to $N_e^{C_2} y \in \pi_{n\rho}^{C_2}(N_e^{C_2} R)$.

Lemma 3.6. Let R be an $\mathbb{E}_\infty^{C_2}$ -ring spectrum, V be a C_2 -representation and $x \in \pi_V^{C_2}(R)$. Let $\deg \tau$ denote the degree of the C_2 -action τ on S^V . Then,

$$\mathrm{Res}_e^{C_2} \mathrm{Nm}_e^{C_2} \mathrm{Res}_e^{C_2} x = (\deg \tau) (\mathrm{Res}_e^{C_2} x)^2 \in \pi_{2|V|}^e(R).$$

Proof. Let $\tau \in C_2$ denote the non-identity element. By the norm-restriction double coset formula⁷ we compute

$$\mathrm{Res}_e^{C_2} \mathrm{Nm}_e^{C_2} (\mathrm{Res}_e^{C_2} x) = (\mathrm{Res}_e^{C_2} x) \cdot \tau_* (\mathrm{Res}_e^{C_2} x) \in \pi_{2|V|}^e(R).$$

So we have to show $\tau_* (\mathrm{Res}_e^{C_2} x) = \mathrm{Res}_e^{C_2} x$. Because $\mathrm{Res}_e^{C_2} x: \mathbb{S}^{|V|} \rightarrow \mathrm{Res}_e^{C_2} R$ is restricted from the C_2 -equivariant map x , there is a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^{|V|} & \xrightarrow{\mathrm{Res}_e^{C_2} x} & R \\ \tau \downarrow & & \downarrow \tau \\ \mathbb{S}^{|V|} & \xrightarrow{\mathrm{Res}_e^{C_2} x} & R \end{array}$$

This diagram encompasses $\tau_* (\mathrm{Res}_e^{C_2} x) = \tau \circ \mathrm{Res}_e^{C_2} x = \mathrm{Res}_e^{C_2} x \circ \tau$, which on homotopy groups is $(\deg \tau) \mathrm{Res}_e^{C_2} x$. \square

Corollary 3.7. Let R be a strongly even $\mathbb{E}_\infty^{C_2}$ -ring spectrum and $n \in \mathbb{Z}$. Suppose that $x \in \pi_{n\rho}^{C_2}(R)$. Then, $\mathrm{Nm}_e^{C_2} \mathrm{Res}_e^{C_2} x = (-1)^n x^2 \in \pi_{2n\rho}^{C_2}(R)$.

Proof. Combine the previous lemma (Lemma 3.6) with the strongly even property, i.e. that the restriction map $\mathrm{Res}_e^{C_2}: \pi_{2n\rho}^{C_2}(R) \rightarrow \pi_{4n}^e(R)$ is an isomorphism. \square

⁷In general for $H \leq G$ it says $\mathrm{Res}_H^G \mathrm{Nm}_H^G a = \prod_{g \in H \backslash G/H} \mathrm{Nm}_{H^g \cap H}^H g_* \mathrm{res}_{H \cap H^g}^H a$. This formula vastly simplifies for $e \leq C_2$.

Proposition 3.8. The C_2 -spectra $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}]$ and $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$ admit $\mathbb{E}_\infty^{C_2}$ -algebra structures.

Proof. Let us only discuss $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$, the same argument works for $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}]$. Let us first note that $\Sigma_+^\infty \text{BU}_\mathbb{R}$ admits an $\mathbb{E}_\infty^{C_2}$ -algebra structure since $\text{BU}_\mathbb{R} \simeq \Omega^\infty \Sigma^\rho \text{ku}_\mathbb{R}$ does. Now, let $\underline{S} \subseteq \pi_\star(\Sigma_+^\infty \text{BU}_\mathbb{R})$ be the smallest C_2 -subset of the $\text{RO}(C_2)$ -graded C_2 -homotopy groups of $\Sigma_+^\infty \text{BU}_\mathbb{R}$ that is closed under norms and contains $\beta_\mathbb{R} \in \pi_\rho^{C_2}(\Sigma_+^\infty \text{BU}_\mathbb{R})$.

Then, the module localization $\Sigma_+^\infty \text{BU}_\mathbb{R}[\underline{S}^{-1}]$ admits an $\mathbb{E}_\infty^{C_2}$ -algebra structure by [HH14, Proposition 4.9]. We wish to see

$$\Sigma_+^\infty \text{BU}_\mathbb{R}[\underline{S}^{-1}] \simeq \Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}][\underline{S}^{-1}] \simeq \Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}].$$

First, let us write $\beta := \text{Res}_e^{C_2} \beta_\mathbb{R} \in \pi_2^e(\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]) \cong \pi_2(\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta^{-1}])$. Next, let us consider the element $N_e^{C_2} \beta \in \pi_{2\rho}(\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}])$. Since $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$ is strongly even (Proposition 3.4), we deduce $\text{Nm}_e^{C_2} \beta = \beta_\mathbb{R}^2$ by Corollary 3.7.

With this computation we now see that the C_2 -set \underline{S} is given by

$$\underline{S}_{C_2} = \{\beta_\mathbb{R}^i \text{Nm}_e^{C_2}(\beta)^j \mid i, j \geq 0\} \quad \text{and} \quad \underline{S}_e = \{\beta^k \mid k \geq 0\}.$$

All of these elements are already inverted in $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$. \square

Theorem 3.9. There are equivalences

$$\text{KU}_\mathbb{R} \simeq \Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}] \quad \text{and} \quad \text{MUP}_\mathbb{R} \simeq \Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$$

of $\mathbb{E}_\infty^{C_2}$ -algebras resp. of \mathbb{E}_ρ -algebras.

Proof. Since equivalences between strongly even C_2 -spectra are checked underlying [HM17, Lemma 3.4], it suffices to lift the classical Snaith equivalences to C_2 -equivariant ones.

Let us begin with the $\text{KU}_\mathbb{R}$ -Snaith. By the universal property of localization [HH14, Proposition 4.9] and the computation of the multiplicative equivariant closure of $\beta_\mathbb{R}$ in Proposition 3.8, it suffices to write down an $\mathbb{E}_\infty^{C_2}$ -algebra map $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty \rightarrow \text{KU}_\mathbb{R}$, which inverts $\beta_\mathbb{R}$ and refines the classical Snaith map. Such a map can be written down in models, as e.g. done in [Sch26, Construction 4.3]. Since the classical Snaith map inverts β , the lift must invert $\beta_\mathbb{R}$ by strong evenness.

Onto the $\text{MUP}_\mathbb{R}$ -Snaith. By Proposition 3.4 and Proposition 3.8 the assumptions in Theorem 2.19 are satisfied, so any \mathbb{E}_2 -Snaith equivalence $\text{MUP} \rightarrow \Sigma_+^\infty \text{BU}[\beta^{-1}]$ provided by Hahn–Yuan [HY20, Proposition 3.1] lifts uniquely to an \mathbb{E}_ρ -algebra map $\text{MUP}_\mathbb{R} \rightarrow \Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$. \square

Remark 3.10. Our proof in particular shows that every (structured) equivalence $\text{MUP} \rightarrow \Sigma_+^\infty \text{BU}[\beta^{-1}]$ admits a unique lift to the Real setting. Similarly, using [QZ26a, Theorem D], every equivalence $\Sigma_+^\infty \mathbb{C}\mathbb{P}^\infty[\beta^{-1}] \rightarrow \text{KU}$ of spectra lifts uniquely to an equivalence $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}] \rightarrow \text{KU}_\mathbb{R}$ of C_2 -spectra. In fact, this strategy gives an alternative proof of $\text{KU}_\mathbb{R}$ -Snaith by lifting the non-equivariant Snaith map instead of writing down the map using models. The downside is that we are currently not able to obtain lifts of structured maps, so we only recover a non-structured $\text{KU}_\mathbb{R}$ -Snaith.

Remark 3.11. The Real Snaith theorem shows that $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty[\beta_\mathbb{R}^{-1}]$ and $\Sigma_+^\infty \text{BU}_\mathbb{R}[\beta_\mathbb{R}^{-1}]$ are Borel C_2 -spectra.

Let $G \geq C_2$ be a finite group. Applying $N_{C_2}^G$ yields higher group versions of the Real Snaith theorem, yielding a variant of [Hil22, Theorem 3.33].

Corollary 3.12. Let $G \geq C_2$ be a finite group. There are equivalences

$$N_{C_2}^G \text{KU}_\mathbb{R} \simeq \Sigma_+^\infty \text{Map}_{C_2}(G, \mathbb{C}\mathbb{P}_\mathbb{R}^\infty)[N_{C_2}^G(\beta_\mathbb{R})^{-1}] \quad \text{and} \quad \text{MUP}^{(G)} \simeq \Sigma_+^\infty \text{Map}_{C_2}(G, \text{BU}_\mathbb{R})[N_{C_2}^G(\beta_\mathbb{R})^{-1}]$$

of \mathbb{E}_∞^G -algebras resp. of $\text{Coind}_{C_2}^G \mathbb{E}_\rho$ -algebras.

Remark 3.13.

- (i) Gepner–Snaith [GS09]⁸ resp. Annala–Iwasa [AI25, Proposition 4.3.3] and Annala–Hoyois–Iwasa [AHI25, Theorem 9.3] provided motivic analogues of the Snaith equivalences, so the Real Snaith theorems without any equivariant multiplicative structures were already implicitly known by C_2 -Betti realization. Schwede had furthermore already generalized the KU-Snaith to an equivalence of ultracommutative Real global spectra before us, thereby also providing a much more structured result. It will appear in forthcoming work of his, as e.g. was announced in [Sch26, Construction 4.3]. His student Song is currently working on a Real global analog of MUP-Snaith.

Nonetheless, our strategy via strong evenness provides a shortcut in the (non-global) Real setting. In particular, the \mathbb{E}_ρ -structure of the $\text{MUP}_{\mathbb{R}}$ -equivalence is novel and demonstrates the versatility of our lifting results [QZ26a].

- (ii) If $\Sigma_+^\infty \text{BU}[\beta^{-1}] \simeq \text{MUP}$ lifts to an \mathbb{E}_4 -equivalence as conjectured by Hahn–Yuan [HY20, Question 3], then our technique would allow us to lift this to an $\mathbb{E}_{2\rho}$ -equivalence.

Applying flavours of C_2 -fixed points to the Real Snaith theorem yields real Snaith theorems. Let us spell out these results. Taking C_2 -geometric fixed points turns the Real Bott element (Construction 3.1) into the Hopf element $\eta: S^1 \rightarrow \mathbb{R}\mathbb{P}^\infty$ resp. $\eta: S^1 \rightarrow \mathbb{R}\mathbb{P}^\infty \rightarrow \text{BO}$.

The following is certainly known classically, but we use this opportunity to demonstrate an application of the Real Snaith theorem. It will afterwards appear in our description of the real Snaith theorems.

Lemma 3.14. There is an equivalence $\Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty[\eta^{-1}] \simeq 0$.

Proof. Applying Φ^{C_2} to the $\text{KU}_{\mathbb{R}}$ -Snaith (Theorem 3.9) yields $\Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty[\eta^{-1}] \simeq \Phi^{C_2} \text{KU}_{\mathbb{R}} \simeq 0$. \square

Corollary 3.15. There are equivalences

$$\text{KO} \simeq \Sigma_+^\infty \mathbb{C}\mathbb{P}_{\mathbb{R}}^\infty[\beta_{\mathbb{R}}^{-1}]_{hC_2} \quad \text{and} \quad \text{MOP} \simeq \Sigma_+^\infty \text{BO}[\eta^{-1}].$$

The latter enhances to an \mathbb{E}_1 -equivalence.

Proof. The second equivalence follows from the Real Snaith theorem (Theorem 3.9) after applying geometric fixed points Φ^{C_2} , and the \mathbb{E}_1 -structure comes from the \mathbb{E}_ρ -structure. For the first equivalence, applying $(-)^{C_2}$ to the Real Snaith theorem (Theorem 3.9) and tom Dieck splitting first yields

$$\text{KO} \simeq \Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty[\eta^{-1}] \oplus \Sigma_+^\infty \mathbb{C}\mathbb{P}_{\mathbb{R}}^\infty[\beta_{\mathbb{R}}^{-1}]_{hC_2}.$$

On the other hand, $\Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty[\eta^{-1}] \simeq 0$ by Lemma 3.14. \square

Remark 3.16. Similarly, Theorem 3.9 with tom Dieck splitting gives rise to a formula for $\text{MUP}_{\mathbb{R}}^{C_2}$ as well, namely $\text{MUP}_{\mathbb{R}}^{C_2} \simeq \Sigma_+^\infty \text{BO}[\eta^{-1}] \oplus \Sigma_+^\infty \text{BU}_{\mathbb{R}}[\beta_{\mathbb{R}}^{-1}]_{hC_2}$.

3.2 Norm inverted Real Snaith theorems

In preparation for THR computations (Theorem 4.7) we will now use the Real Snaith theorems (Theorem 3.9) to obtain an alternate Snaith construction. Let us begin by recording a (certainly well-known) classical fact about localizations.

Lemma 3.17. Let R be an \mathbb{E}_∞ -ring spectrum and $n \in \mathbb{N}$ with $x, \varepsilon \in \pi_n R$. Suppose that ε is nilpotent in $\pi_* R$. Then, the localization map $R \rightarrow R[(x + \varepsilon)^{-1}]$ induces an equivalence $R[x^{-1}] \xrightarrow{\simeq} R[(x + \varepsilon)^{-1}]$ of \mathbb{E}_∞ -ring spectra.

Proof. First, we need to show that x is invertible in $R[(x + \varepsilon)^{-1}]$ so that the localization map induces a map $R[x^{-1}] \rightarrow R[(x + \varepsilon)^{-1}]$. We see this by writing $x = (x + \varepsilon) \left(1 - \frac{\varepsilon}{x + \varepsilon}\right)$, which is a product of two units: $x + \varepsilon$ is invertible in $R[(x + \varepsilon)^{-1}]$ by construction and the second term is the sum of a unit and a nilpotent element and thus invertible.

Conversely, $x + \varepsilon$ as a sum of a unit and a nilpotent element is invertible in $R[x^{-1}]$, so the localization map $R \rightarrow R[x^{-1}]$ induces a map $R[(x + \varepsilon)^{-1}] \rightarrow R[x^{-1}]$.

⁸Beware that there is a mistake in this paper, see [AHI25, Footnote 9].

Composing these two maps yields two maps

$$R[x^{-1}] \longrightarrow R[x^{-1}] \quad \text{and} \quad R[(x + \varepsilon)^{-1}] \longrightarrow R[(x + \varepsilon)^{-1}],$$

both induced by the localization map. Thus, these must be the identities, which means that we have constructed inverse maps. \square

Lemma 3.18. The C_2 -spectra $\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty [\text{Nm}_e^{C_2}(\beta)^{-1}]$ and $\Sigma_+^\infty \text{BU}_\mathbb{R} [\text{Nm}_e^{C_2}(\beta)^{-1}]$ admit $\mathbb{E}_\infty^{C_2}$ -algebra structures such that there are equivalences

$$\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty [\text{Nm}_e^{C_2}(\beta)^{-1}] \simeq \Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty [\beta_\mathbb{R}^{-1}] \quad \text{and} \quad \Sigma_+^\infty \text{BU}_\mathbb{R} [\text{Nm}_e^{C_2}(\beta)^{-1}] \simeq \Sigma_+^\infty \text{BU}_\mathbb{R} [\beta_\mathbb{R}^{-1}]$$

of $\mathbb{E}_\infty^{C_2}$ -algebras.

Proof. There are maps

$$\Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty [\text{Nm}_e^{C_2}(\beta)^{-1}] \longrightarrow \Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty [\beta_\mathbb{R}^{-1}] \quad \text{and} \quad \Sigma_+^\infty \text{BU}_\mathbb{R} [\text{Nm}_e^{C_2}(\beta)^{-1}] \longrightarrow \Sigma_+^\infty \text{BU}_\mathbb{R} [\beta_\mathbb{R}^{-1}]$$

of \mathbb{E}_∞ -algebras, and it suffices to show that these are equivalences. This then shows in particular, that the sources obtain $\mathbb{E}_\infty^{C_2}$ -algebra structures by Hill–Hopkins [HH14]. We do this by checking it on geometric fixed points. Since $\text{Res}_e^{C_2} \text{Nm}_e^{C_2}(\beta) = \beta^2$ by Lemma 3.6, the map is evidently an equivalence after applying Φ^e . So let us discuss Φ^{C_2} .

Let us start with the $\mathbb{C}\mathbb{P}_\mathbb{R}^\infty$ -version. First, we observe $\Phi^{C_2} \Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty [\beta_\mathbb{R}^{-1}] \simeq \Phi^{C_2} \text{KU}_\mathbb{R} \simeq 0$ by the Real Snaith theorem (Theorem 3.9). We need to compare it to $\Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty [(r\beta)^{-1}]$ where $r\beta = \Phi^{C_2} \text{Nm}_e^{C_2}(\beta)$ is the element

$$r\beta: \mathbb{S}^2 \simeq \Phi^{C_2} N_e^{C_2} \mathbb{S}^2 \xrightarrow{\beta} \Phi^{C_2} N_e^{C_2} \Sigma_+^\infty \mathbb{C}\mathbb{P}^\infty \xrightarrow{r} \Phi^{C_2} \Sigma_+^\infty \mathbb{C}\mathbb{P}_\mathbb{R}^\infty \simeq \Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty.$$

In other words, we need to show that $r\beta \in \pi_2(\Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty)$ is nilpotent. We achieve this with the Devinatz–Hopkins–Smith nilpotence theorem [DHS88].⁹ Indeed, $\pi_2(\Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty)$ is torsion since $H_2(\Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty; \mathbb{Q}) = 0$, but $\text{MU}_2(\Sigma_+^\infty \mathbb{R}\mathbb{P}^\infty) \cong \mathbb{Z}$.¹⁰ This is torsionfree. So $r\beta$ lies in the kernel of the Hurewicz map of MU and thus is nilpotent by the nilpotence theorem.

Now onto the $\text{BU}_\mathbb{R}$ -version. We consider the two elements $r\beta = \Phi^{C_2} \text{Nm}_e^{C_2}(\beta)$ and $\eta^2 = \Phi^{C_2} \beta_\mathbb{R}^2$ defined as follows:¹¹

$$\begin{aligned} r\beta: \mathbb{S}^2 &\simeq \Phi^{C_2} N_e^{C_2} \mathbb{S}^2 \xrightarrow{\beta} \Phi^{C_2} N_e^{C_2} \Sigma_+^\infty \text{BU}_\mathbb{R} \xrightarrow{r} \Phi^{C_2} \Sigma_+^\infty \text{BU}_\mathbb{R} \simeq \Sigma_+^\infty \text{BO}, \\ \eta^2: \mathbb{S}^2 &\simeq \Phi^{C_2} \mathbb{S}^{2\rho} \xrightarrow{\beta^2} \Phi^{C_2} \Sigma_+^\infty \text{BU}_\mathbb{R} \simeq \Sigma_+^\infty \text{BO}. \end{aligned}$$

Thus, we need to show that the map of \mathbb{E}_∞ -ring spectra

$$\Sigma_+^\infty \text{BO}[(r\beta)^{-1}] \longrightarrow \Sigma_+^\infty \text{BO}[\eta^{-2}]$$

is an equivalence. By Lemma 3.17, it suffices to prove that $r\beta - \eta^2$ is nilpotent. We wish to employ May nilpotence [MNN15]. So consider the Hurewicz homomorphism

$$h: \pi_2(\Sigma_+^\infty \text{BO}) \longrightarrow H_2(\text{BO}; \mathbb{Z}).$$

A Serre spectral sequence computation combined with the universal coefficient theorem shows that there is an isomorphism $H_2(\text{BO}; \mathbb{Z}) \cong \mathbb{Z}/2$. Since $\Sigma_+^\infty \text{BO}[\eta^{-2}] \simeq \text{MOP} \neq 0$ by Corollary 3.15, the element η^2 cannot be nilpotent, so $h(\eta^2) = 1$ by May nilpotence. On the other hand, we have a map of \mathbb{E}_∞ -ring spectra $\Sigma_+^\infty \text{BO}[(r\beta)^{-1}] \rightarrow \Sigma_+^\infty \text{BO}[\eta^{-2}]$ with non-zero target, so the source must also be non-zero. Hence, $r\beta$ is not nilpotent, whence $h(r\beta) = 1$ by May nilpotence. By linearity, we conclude

$$h(r\beta - \eta^2) = h(r\beta) - h(\eta^2) = 1 - 1 = 0,$$

so $r\beta - \eta^2 \in \pi_*(\Sigma_+^\infty \text{BO})$ is nilpotent by May nilpotence, as desired. \square

⁹One could also have used the May nilpotence theorem.

¹⁰The map $\text{MU} \rightarrow \mathbb{Z}$ is 2-connective, so $\text{fib}(\text{MU} \rightarrow \mathbb{Z}) \otimes \Sigma^\infty \mathbb{R}\mathbb{P}^\infty$ is 3-connective. By the associated LES the map $\text{MU}_2(\Sigma^\infty \mathbb{R}\mathbb{P}^\infty) \rightarrow H_2(\Sigma^\infty \mathbb{R}\mathbb{P}^\infty) \cong 0$ is an isomorphism.

¹¹In particular, we abuse notation and use the notation $r\beta$ again as in the $\mathbb{C}\mathbb{P}_\mathbb{R}^\infty$ -version.

Combining this with the Real Snaith theorems (Theorem 3.9) immediately yields:

Theorem 3.19. There are equivalences

$$\mathrm{KU}_{\mathbb{R}} \simeq \Sigma_+^{\infty} \mathbb{C}\mathbb{P}_{\mathbb{R}}^{\infty} [\mathrm{Nm}_e^{C_2}(\beta)^{-1}] \quad \text{and} \quad \mathrm{MUP}_{\mathbb{R}} \simeq \Sigma_+^{\infty} \mathrm{BU}_{\mathbb{R}} [\mathrm{Nm}_e^{C_2}(\beta)^{-1}]$$

of $\mathbb{E}_{\infty}^{C_2}$ - resp. \mathbb{E}_{ρ} -algebras.

Taking $(-)^{C_2}$ and applying tom Dieck splitting as in Corollary 3.15 yields a norm inverted version of the real Snaith theorem for KO:

Corollary 3.20. There is an equivalence $\mathrm{KO} \simeq \Sigma_+^{\infty} \mathbb{C}\mathbb{P}_{\mathbb{R}}^{\infty} [\mathrm{Nm}_e^{C_2}(\beta)^{-1}]_{hC_2}$ of \mathbb{E}_{∞} -algebras.

4 Computations in Real topological Hochschild homology

4.1 Structured Real Bökstedt equivalence & Inverting elements in THR

We will now give applications to THR-computations. Recall that for an \mathbb{E}_{σ} -algebra R one can define $\mathrm{THR}(R) := R \otimes_{N_e^{C_2} R_e} R$, which for $\mathbb{E}_{\infty}^{C_2}$ -algebras R can be described as $\mathrm{THR}(R) \simeq R \otimes S^{\sigma}$, see [DMPR21] and [QS22, Remark 5.3].

Our first THR-computation concerns Real Bökstedt periodicity. Bayındır–Moulinos showed that Bökstedt periodicity $\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Sigma_+^{\infty} \Omega S^3$ refines to an \mathbb{E}_2 -equivalence [BM22, Theorem 1.3], and we lift it to an \mathbb{E}_{ρ} -equivalence. We include it here to demonstrate that our strategy to prove the Real Snaith theorems is versatile enough to be applicable in multiple settings.

Theorem 4.1. There is an equivalence $\mathrm{THR}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Sigma_+^{\infty} \Omega S^{1+\rho}$ of \mathbb{E}_{ρ} -algebras.

Proof. It is known that there is such an equivalence of C_2 -spectra [DMPR21, Theorem 5.18], which in particular implies that it is strongly even. Since \mathbb{F}_p is naturally $\mathbb{E}_{\infty}^{C_2}$, it induces an \mathbb{F}_p - $\mathbb{E}_{\infty}^{C_2}$ -algebra structure on $\mathrm{THR}(\mathbb{F}_p)$, see Observation 4.2. Moreover, $S^{1+\rho}$ admits an \mathbb{E}_{σ} -algebra structure, since there is an equivalence $S^{1+\rho} \simeq \Omega^{\sigma} \mathbb{H}\mathbb{P}_{\mathbb{R}}^{\infty}$ by [HS20, Proposition 4.2],¹² so the other side of the equivalence admits an \mathbb{F}_p - \mathbb{E}_{ρ} -algebra structure as well.

Now, \mathbb{F}_p - \mathbb{E}_{ρ} -maps $\mathbb{F}_p \otimes \Sigma_+^{\infty} \Omega S^{1+\rho} \rightarrow \mathrm{THR}(\mathbb{F}_p)$ correspond to maps

$$\Sigma^{\infty} \mathbb{H}\mathbb{P}_{\mathbb{R}}^{\infty} \simeq \Sigma^{\infty} \mathrm{B}^{\sigma} S^{1+\rho} \longrightarrow \mathrm{gl}_1 \mathrm{THR}(\mathbb{F}_p)$$

of C_2 -spectra by an adjunction argument, see e.g. [QZ26a, Proposition 5.4.1]. On underlying, there is a map which corresponds to an \mathbb{E}_2 -equivalence $\mathrm{THH}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes \Sigma_+^{\infty} \Omega S^3$ by Bayındır–Moulinos [BM22, Theorem 1.3]. It suffices to lift this map since equivalences of strongly even spectra can be checked underlying [HM17, Lemma 3.4]. To do this, we want to apply [QZ26a, Theorem 5.3.8], for which the only missing assumption is that $\Sigma^{\infty} \mathbb{H}\mathbb{P}_{\mathbb{R}}^{\infty}$ has finite type ρ -free homology.

To see this, we consider a C_2 -cell decomposition of $\mathbb{H}\mathbb{P}_{\mathbb{R}}^{\infty}$ suggested by Mike Hopkins and spelled out by Hahn–Shi [HS20, Proposition 4.2] – namely one with a single $2n\rho$ -cell for each $n \geq 0$. Thus, $\Sigma_+^{\infty} \mathbb{H}\mathbb{P}_{\mathbb{R}}^{\infty}$ has finite type ρ -free homology analogously to [Hil22, Section 4.3.1], see e.g. [PRS21, Example 5.2]. \square

Next, we discuss the interaction of THR with inversions. With it we set up the main ingredient in computing $\mathrm{THR}(\mathrm{KU}_{\mathbb{R}})$ and $\mathrm{THR}(\mathrm{MUP}_{\mathbb{R}})$ using the Real Snaith theorems (Theorem 3.9), see Theorem 4.7.

Observation 4.2. Let R be an $\mathbb{E}_{\infty}^{C_2}$ -ring spectrum. Then, the counit $N_e^{C_2} \mathrm{Res}_e^{C_2} R \rightarrow R$ is an $\mathbb{E}_{\infty}^{C_2}$ -algebra map. In particular, the map

$$R \simeq N_e^{C_2} \mathrm{Res}_e^{C_2} R \otimes_{N_e^{C_2} \mathrm{Res}_e^{C_2} R} R \longrightarrow R \otimes_{N_e^{C_2} \mathrm{Res}_e^{C_2} R} R \simeq \mathrm{THR}(R)$$

is an $\mathbb{E}_{\infty}^{C_2}$ -algebra map, thus endowing $\mathrm{THR}(R)$ with an R -module structure. This gives sense to invert homotopy group elements from R in $\mathrm{THR}(R)$ in an R -module fashion.

Both $N_e^{C_2}$ and $\mathrm{Nm}_e^{C_2}$ will show up in the proof of the following statement, so we refer the reader to Notation 3.5 for a reminder of this notation. Our remark ensures that the objects in the statement are well-defined.

¹²The C_2 -action on $\mathbb{H}\mathbb{P}^{\infty}$ is given by conjugation by i .

Proposition 4.3. Let R be an $\mathbb{E}_\infty^{C_2}$ -ring spectrum and V be a C_2 -representation. Suppose that $\bar{x} \in \pi_V^{C_2}(R)$ and $x = \text{Res}_e^{C_2} \bar{x} \in \pi_{|V|}^e(R)$. Then, there is an equivalence

$$\text{THR}\left(R[(\text{Nm}_e^{C_2} x)^{-1}]\right) \simeq \text{THR}(R)[(\text{Nm}_e^{C_2} x)^{-1}]$$

of $\mathbb{E}_\infty^{C_2}$ -algebras.

Proof. Let us first note that both sides admit $\mathbb{E}_\infty^{C_2}$ -algebra structures by Hill–Hopkins [HH14], see also our argument in Proposition 3.8.

Now, we can compute

$$\begin{aligned} \text{THR}\left(R[(\text{Nm}_e^{C_2} x)^{-1}]\right) &\simeq R[(\text{Nm}_e^{C_2} x)^{-1}] \otimes_{N_e^{C_2} \text{Res}_e^{C_2} R[(\text{Nm}_e^{C_2} x)^{-1}]} R[(\text{Nm}_e^{C_2} x)^{-1}] \\ &\simeq R[(\text{Nm}_e^{C_2} x)^{-1}] \otimes_{(N_e^{C_2} \text{Res}_e^{C_2} R)[(\text{Nm}_e^{C_2} x)^{-1}]} R[(\text{Nm}_e^{C_2} x)^{-1}]. \end{aligned}$$

On the other hand, $R[(\text{Nm}_e^{C_2} x)^{-1}] \simeq (N_e^{C_2} \text{Res}_e^{C_2} R)[(N_e^{C_2} x)^{-1}] \otimes_{(N_e^{C_2} \text{Res}_e^{C_2} R)} R$, so this term simplifies into

$$R[(\text{Nm}_e^{C_2} x)^{-1}] \otimes_{(N_e^{C_2} \text{Res}_e^{C_2} R)} R \simeq \text{THR}(R)[(\text{Nm}_e^{C_2} x)^{-1}],$$

as desired. \square

Remark 4.4. The object

$$(N_e^{C_2} \text{Res}_e^{C_2} R)[(N_e^{C_2} x)^{-1}] \otimes_{(N_e^{C_2} \text{Res}_e^{C_2} R)} R \simeq N_e^{C_2} \text{Res}_e^{C_2} (R[(\text{Nm}_e^{C_2} x)^{-1}]) \otimes_{(N_e^{C_2} \text{Res}_e^{C_2} R)} R$$

appearing in the above proof is also known as $N_R^{e \rightarrow C_2}(R[(\text{Nm}_e^{C_2} x)^{-1}])$, the norm in the C_2 -symmetric monoidal ∞ -category $\underline{\text{LMod}}_R$, see e.g. [QZ26a, Example 3.3.14].

Corollary 4.5. Let R be a strongly even $\mathbb{E}_\infty^{C_2}$ -ring spectrum and $n \in \mathbb{Z}$. Consider $\bar{x} \in \pi_{n\rho}^{C_2}(R)$. Then, there is an equivalence

$$\text{THR}(R[\bar{x}^{-1}]) \simeq \text{THR}(R)[\bar{x}^{-1}]$$

of $\mathbb{E}_\infty^{C_2}$ -ring spectra.

Proof. By Corollary 3.7 we have $\text{Nm}_e^{C_2} \text{Res}_e^{C_2} \bar{x} = (-1)^n \bar{x}^2$, but inverting $(-1)^n \bar{x}^2$ is the same thing as inverting \bar{x} . Moreover, one can now employ the same argument as in Proposition 3.8 to obtain $\mathbb{E}_\infty^{C_2}$ -ring structures on the localizations. \square

We can recover the classical result about commuting THH with inverting elements [Sto20, Corollary 4.12].

Corollary 4.6. Let R be an \mathbb{E}_∞ -ring spectrum, then we can recover the classical equivalence

$$\text{THH}(R[x^{-1}]) \simeq \text{THH}(R)[x^{-1}]$$

for every $x \in \pi_* R$.

Proof. Endowed with the trivial action, the Borelification of R becomes an $\mathbb{E}_\infty^{C_2}$ -ring spectrum by [HM17, Theorem 2.4, 2.7]. Now, consider the element $\bar{x} = \text{Nm}_e^{C_2} x$. Then, $\text{Res}_e^{C_2} \bar{x} = x \cdot \tau_* x = x^2$ by the norm-restriction double coset formula remembering that R has trivial C_2 -action. Since inverting x^2 is equivalent to inverting x , the underlying of Proposition 4.3 yields the THH statement. \square

4.2 Applications of Real Snaith in THR computations

Our first THR-result applies the Real Snaith theorem. Stonek showed $\text{THH}(\text{KU}) \simeq \text{KU}[\text{B}^2\text{U}(1)]$ using Snaith’s theorem, see [Sto20, Theorem 5.21] and [RSV22, Example 7.1]. Using the Real Snaith theorem, we can lift this equivalence.

Theorem 4.7. There are equivalences

$$\text{THR}(\text{KU}_\mathbb{R}) \simeq \text{KU}_\mathbb{R} \otimes \Sigma_+^\infty \text{B}^\rho \text{U}_\mathbb{R}(1) \quad \text{resp.} \quad \text{THR}(\text{MUP}_\mathbb{R}) \simeq \text{MUP}_\mathbb{R} \otimes \Sigma_+^\infty \text{B}^\rho \text{U}_\mathbb{R}$$

of $\mathbb{E}_\infty^{C_2}$ -algebras resp. of \mathbb{E}_1 -algebras.

Proof. Let us demonstrate the computation for $\mathrm{THR}(\mathrm{MUP}_{\mathbb{R}})$, the same works for $\mathrm{THR}(\mathrm{KU}_{\mathbb{R}})$. By the norm inverted Real Snaith theorem ([Theorem 3.19](#)) we have $\mathrm{THR}(\mathrm{MUP}_{\mathbb{R}}) \simeq \mathrm{THR}(\Sigma_+^{\infty} \mathrm{BU}_{\mathbb{R}} [\mathrm{Nm}_e^{C_2}(\beta)^{-1}])$. Using [Proposition 4.3](#) we may commute the localization with THR . Combined with the computation of THR of spherical group rings [[DMPR21](#), Proposition 5.12], we obtain

$$\begin{aligned} \mathrm{THR}(\mathrm{MUP}_{\mathbb{R}}) &\simeq \mathrm{THR}(\Sigma_+^{\infty} \mathrm{BU}_{\mathbb{R}} [\mathrm{Nm}_e^{C_2}(\beta)^{-1}]) \\ &\simeq (\Sigma_+^{\infty} \mathrm{BU}_{\mathbb{R}} \otimes \Sigma_+^{\infty} \mathrm{B}^{\rho} \mathrm{U}_{\mathbb{R}}) [\mathrm{Nm}_e^{C_2}(\beta)^{-1}] \\ &\simeq \mathrm{MUP}_{\mathbb{R}} \otimes \Sigma_+^{\infty} \mathrm{B}^{\rho} \mathrm{U}_{\mathbb{R}}, \end{aligned}$$

as desired. □

Applying Φ^{C_2} yields:

Corollary 4.8. There is an \mathbb{E}_1 -equivalence $\mathrm{MOP} \otimes_{\mathrm{MUP}} \mathrm{MOP} \simeq \mathrm{MOP} \otimes \Sigma_+^{\infty} \mathrm{B}^{\rho} \mathrm{U}_{\mathbb{R}}^{C_2}$.

Remark 4.9. Using techniques to compute THR of Thom spectra via factorization homology Hovey–Klang–Zou compute $\mathrm{THR}(\mathrm{MU}_{\mathbb{R}}) \simeq \mathrm{MU}_{\mathbb{R}} \otimes \Sigma_+^{\infty} \mathrm{B}^{\rho} \mathrm{U}_{\mathbb{R}}$, see [[HHK⁺24](#), Corollary 7.2.2]. This is however conditional on a G -symmetric monoidal straightening-unstraightening result [[QZ26a](#), Remark 7.1.2], which is not yet recorded in the literature. We will discuss such a result in forthcoming work with Siddharth.

We note that the formula for the THR of Thom spectra [[HHK⁺24](#), Corollary 7.1.3] requires that the base space of the Thom spectrum is connected, so it cannot be used to compute $\mathrm{THR}(\mathrm{MUP}_{\mathbb{R}})$. Even if one can get rid of the connectedness assumption, it would yield $\mathrm{THR}(\mathrm{MUP}_{\mathbb{R}}) \simeq \mathrm{MUP}_{\mathbb{R}} \otimes \Sigma_+^{\infty} \mathrm{B}^{\sigma} \mathrm{BUP}_{\mathbb{R}}$, which is different from our computation in [Theorem 4.7](#). This is in parallel to the non-equivariant phenomenon

$$\mathrm{THH}(\mathrm{MUP}) \simeq \mathrm{MUP} \otimes \Sigma_+^{\infty} \mathrm{SU} \simeq \mathrm{MUP} \otimes \Sigma_+^{\infty} \mathrm{U}$$

using Snaith resp. the THH formula for Thom spectra. See [[RSV22](#), Example 7.13].

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